For the use of exterior form in daily physics, an introduction without coordinate frame.

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July 13, 2023

This is a short introduction of the exterior form formalism focus on its applications in physics and then mostly aimed to physics students. As a rule of a game played here we never use a coordinate frame neither in the definitions nor in the proofs but only at the end in order to recover the classical physics equations. This approach is unusual but we think is helpful for the understanding and very valuable to grab the physical meanings of the mathematical object.

A large part of these notes are just "notations rewriting" of well known physical objects but this should not be underestimate as it gives short and elegant expressions that are useful both for insights and computations. Appart from the *game* explained above most the material presented here is very well known [Fra11, Fla63, BBB85] with exeption maybe of Corollary 11 for which we have surprisingly not found its statement in the litterature and the discussion around Corollaries 7, 8 which we believe deserves more publicity.

1 Exterior forms

1.1 Integration on a submanifold

Usually, introduction about exterior form start at the coordinate level giving the formal definition of an exterior algebra [Pau07, God70, LCC$^+$09]. However, it is not so obvious at first sight why such an object is usefull and interesting in physics. So we would instead propose the following definition to go immediatly to the main point [Fla63].

Definition 1. A $k-$form is *what* to be integrated on a $k-$dimension submanifold.

This definition is made vague and unformal on purpose. Because it much closer of what any physics would want as a mathematical object. That is to choose an submanifold[1] : a path, a surface, a volume or a time interval $\times$ a volume and have physical quantity to integrate on it. From the lowest mathematical level this is just an application α that for any $k-$dimension submanifold

[1] More precisely an "oriented submanifold" as the integration on path from a to b or from b to a will have a different sign

$\mathcal{V}$ associate a real number $\alpha(\mathcal{V}) \in \mathbb{R}$. This should really be though as the "meaningful physical quantity", as it doesn't depend on the choice of coordinate, not even the space-time metric g. If one think of a glass of water the quantity of water in the volume defined by the glass (an 3 dimension submanifold) should exist and be independant on the methods we measure space or time. One would denote

$$\alpha(\mathcal{V}) = \int_{\mathcal{V}} \alpha.$$

Here are some examples.

quantity	dimension	basis	unit
temperature gradient	1-form	dx, dy, dz	$K.\boldsymbol{m}^{-1}$
radiation	2-form	$dx \wedge dy, dx \wedge dz, dy \wedge dz$	$W.\boldsymbol{m}^{-2}$
density of mass	3-form	$dx \wedge dy \wedge dz$	$kg.\boldsymbol{m}^{-3}$
flow	3-form	$dx \wedge dy \wedge dt, ...$	$kg.\boldsymbol{m}^{-2}\boldsymbol{s}^{-1}$
Field Lagrangian	4-form	$dx \wedge dy \wedge dz \wedge dt$	$J.\boldsymbol{m}^{-3}\boldsymbol{s}^{-1}$

Notice that in $\mathbb{R}^3$ the dimension of the $k-$form $\Lambda^k(\mathbb{R}^3)$ are respectively $1, 3, 3$ et 1. The 0-form and 3-form are usually seen as "scalar" whereas 1-form and 2-form are usually seen as "vector field". In $\mathbb{R}^4$ (or higher dimension) there is no such equivalence for the 2-form. But even in $\mathbb{R}^3$ it is still interesting to distinguish them at the mathematical level since they would not behave the same way with a change of coordinate. A fact that stills appears in the system of units.

1.2 Tangent vector fields

"Vector field" is also usually used to design another mathematical object that to avoid ambiguity we will call here *tangent vector fields.* If exterior forms are associated with integration, one intead should think of tangent vector fields [LCC+09] informally as follow.

Definition 2. A tangent vector field is *what* describes a flow, ie a transport.

More precisely for X a vector field and a point x_0 we have a natural associated flow $x(t)$ solution of the equation

$$\partial_t x(t) = X(x(t)) \qquad x(0) = x_0. \tag{1}$$

For example

quantity	object	basis	unit
velocity of a fluid	tangent vector	$\partial_x, \partial_y, \partial_z$	$\boldsymbol{m}.s^{-1}$

One can use tangent vector fields to define different operations on exterior forms. But because this is not completly obvious we first define a few opérations on a submanifold $\mathcal{V}$:

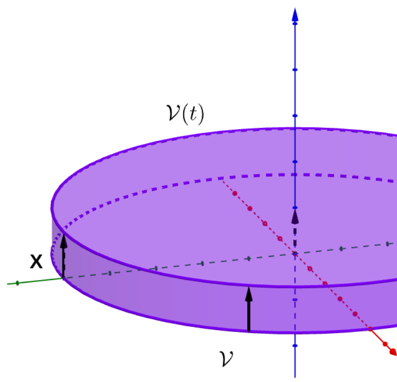

Figure 1: Here X is the vertical vector field, $\mathcal{V}$ is the disk bellow, $\partial\mathcal{V}$ is the circle bellow, $\mathcal{V}(t)$ is disk above, $\partial\mathcal{V}(t)$ is the circle above, $\mathcal{I}_t(\mathcal{V})$ is the cylinder (volume) and $\mathcal{I}_t(\partial\mathcal{V})$ is the ring around the cylinder.

- The transport of the submanifold $\mathcal{V}(t) = \{x(t), x_0 \in \mathcal{V}\}$. One can think for example as a blade of grass or a leave transported in the flow of water.
- An enlarged submanifold $\mathcal{I}_t(\mathcal{V})$ that is a submanifold which is one dimensional larger than $\mathcal{V}$ defined as the union of the $\mathcal{V}(s)$ for $s \in [0, t]$. We can see it as keeping track of the successive positions of transported manifold along the flow as in an numerical simulation. For example in the case of a point $\mathcal{V} = \{x_0\}$, this is just the line traced by the trajectory $x(t)$. In the case of the leave it is the volume in the water that has been covered by the leave on its way.

Both $\mathcal{V}(t)$ and $\mathcal{I}_t(\mathcal{V})$ are natural objects on which integrate an exterior form.

1.3 A few elementary operations on exterior forms

We now present some basic operations to manipulate exterior form as appears in math textbooks [JJ08, LCC$^+$09, Fra11]. Maybe this section is more aimed to students in mathematics but it fit in the "no coordinate approach" for definitions and proofs of this introduction. Also the operations presented here have some physical meaning that we would like to stress. The philosophy here is the operations on submanifold can be translated into an operation on form. More precisely consider an application that from a k submanifold $\mathcal{V}$ gives another submanifold $\mathcal{V} \to \tilde{\mathcal{V}}$. As a fondamental rule, we will ask that the "physical quantity" does not change. This defines either an application that from a k form α gives another form $\alpha \to \tilde{\alpha}$ such that

$$\tilde{\alpha}(\tilde{\mathcal{V}}) = \alpha(\mathcal{V}) \tag{2}$$

or a dual application that define α from $\tilde{\alpha}$: $\tilde{\alpha} \to \alpha$ also with (2). Here are some examples with an tangent vector field X.

- *Change of coordinate* : we can describe the submanifold as $\mathcal{V}$ in a coordinate frame or as $\tilde{\mathcal{V}}$ in another coordinate frame. The transformation $\alpha \to \tilde{\alpha}$ given Equation (2) means that α is a "k dimensional covariant tensor". (On the other hand the tangent vector field X is a contravariant tensor).
- The *pullback transport* : Here we use another notation for the flow $\phi_t(x(0)) = x(t)$ as in (1) and define $\phi_t^*(\alpha)$ as a k form such that for any k dimensional submanifold $\mathcal{V}$
$$[\phi_t^*(\alpha)](\mathcal{V}) = \alpha(\mathcal{V}(t)).$$
- For α a k form, we can then introduce a $k-1$ form $I_t\alpha$ defined such that for any $k-1$ dimensional submanifold $\mathcal{V}$
$$[I_t\alpha](\mathcal{V}) = \alpha(\mathcal{I}_t(\mathcal{V})).$$

At first order when $t \to 0$ we have the following "derivative" operations.

- The *Lie derivative*
$$L_X\alpha = \frac{d}{dt}|_{t=0}\phi_t^*(\alpha) = \lim_{t\to 0}\frac{\phi_t^*(\alpha) - \alpha}{t}.$$
- The *interior product*
$$i_X\alpha = \frac{d}{dt}|_{t=0}I_t\alpha = \lim_{t\to 0}\frac{I_t\alpha - \alpha}{t}.$$
This has a natural physical meaning : for example with ρ a density of matter (a 3-form) and X a velocity field, $i_X\rho$ (a 2-form) describes the density flow (For any surface S, $i_X\rho(S)$ is the flow of matter that is going through S).

All these operations also have a simple definition in a coordinate system. However in my opinion it is still important to start from a geometric point of view since it gives an motivation to introduce these objects and also it makes it clear that they are independent of the choice of the coordinate and therefore should have some physical meaning.

2 Exterior Derivative

2.1 Integration on the boundary

The usual formal definition of the exterior derivative use the derivation at the coordinate level [JJ08, LCC+09, Fra11] but as before it is not very clear at first sight why such an objet would be interesting in physics. So here again is another definition that is informal but that looks very natural and capture the main interest of the object.

Definition 3. For a $k-$form α, the exterior derivative $d\alpha$ is the $(k+1)-$form such that $d\alpha(\mathcal{V}) = \alpha(\partial\mathcal{V})$ for any $(k+1)$ dimensional submanifold $\mathcal{V}$.

This is of course the very famous Stokes Theorem.

$$\int_{\mathcal{V}} d\alpha = \int_{\partial\mathcal{V}} \alpha$$

but as in Definition 1 such an equation could also be seen as a definition of an application on the $(k+1)-$submanifolds, that is a $(k+1)-$form and such a definition does not depend on a choice coordinate.

Also boundaries of submanifold are very commun objects : the extremal points of a path, the circle surounding a disc, the surface around a volume. One can also think of the initial and final configurations of a system as the 3 dimension boundaries of the evolving system in time and space seen as a 4 dimension submanifold.

In a 3 dimension space whether the form α is a $0, 1$ or 2-form, the exterior derivative $d\alpha$ is called gradient, curl and divergence[2].

$$\Lambda^0(\mathbb{R}^3) \xrightarrow{\text{grad}} \Lambda^1(\mathbb{R}^3) \xrightarrow{\text{curl}} \Lambda^2(\mathbb{R}^3) \xrightarrow{\text{div}} \Lambda^3(\mathbb{R}^3).$$

where $\Lambda^k(\mathbb{R}^3)$ denote the space of $k-$form.

We also state the following important observation : «the boundary of a manifold has no boundary» $\partial(\partial\mathcal{V}) = \emptyset$. Therefore for any form α, we have $(d \circ d\alpha)(\mathcal{V}) = \alpha(\partial(\partial\mathcal{V})) = 0$ and we write the following proposition.

Proposition 4. $d \circ d = 0$

For example in $\mathbb{R}^3$ this is the well known $\text{curl} \circ \text{grad} = 0$ and $\text{div} \circ \text{curl} = 0$.

2.2 Conserved quantity

With the previous section in $\mathbb{R}^3$ a divergence-free vector field is then a 2-form α such that $d\alpha = 0$. In $\mathbb{R}^4$. we can propose a similar definition of a conserved quantity. If we go back to our example of evolving system in time and space. For conservation we ask that the integration on the initial and final configurations gives the same result. This is guaranty if $d\alpha = 0$ so we can propose the following definition.

Definition 5. A conserved quantity is a $3-$form[3] J such that $dJ = 0$.

Writting the 3-form in a coordinate frame $J = (\rho, j_x, j_y, j_z)$, the condition $dJ = 0$ is just the Continuous Equation [Fla63]

$$\partial_t \rho + \text{div}(j) = 0.$$

[2]The most commun convention use the metric tensor to transform these forms into vector field.

[3]Or $(n-1)-$form if the system is n dimensional

An advantage of the above formulation is that it does not depends on the coordinate frame or the metric. Equivalently for any $\mathcal{V}$ we have $J(\partial\mathcal{V}) = 0$. In word : "What is going into $\mathcal{V}$ is the same as what is going out of $\mathcal{V}$" which is also a very natural non mathematical definition of what a conserved quantity is.

2.3 A few mathematical relations

We go back to the math textbooks [JJ08, LCC+09, Fra11]. Here are some remarks related to the boundaries (see for example Figure 1) :

1. The boundary of the transported submanifold is just the transport of the boundary of the inital submanifold: $\partial[\mathcal{V}(t)] = [\partial\mathcal{V}](t)$. As a consequence for $\alpha = d\beta$ we have

$$\phi_t^*(d\beta) = d\phi_t^*(\beta) \quad \text{and} \quad dL_X\beta = L_X(d\beta).$$

2. The boundary of $\mathcal{I}_t(\mathcal{V})$ has three terms : $\mathcal{V}$, $\mathcal{V}(t)$ and the union of $\partial\mathcal{V}(s)$ for $s \in [0, t]$. The latter also corresponds to $\mathcal{I}_t(\partial\mathcal{V})$. Therefore we have[4]

$$[I_t d\beta](\mathcal{V}) = d\beta(\mathcal{I}_t(\mathcal{V})) = \beta(\mathcal{V}(t)) - \beta(\mathcal{V}(0)) - \beta(\mathcal{I}_t(\partial\mathcal{V}))$$

and then

$$I_t d\beta = \phi_t^*(\beta) - \beta - dI_t\beta \quad \text{and} \quad L_X = d \circ i_X + i_X \circ d.$$

This last relation is known as Cartan's magic formula.

3 Maxwell equations

One of the most beautiful application of the exterior form formalism is that it gives a clean and unified picture of the classical theory of electromagnetism [Fla63, Fra11].

3.1 The metric and the $\star$ operator

From a purely mathematical point of view of the exterior forms the metric appears in a quite indirect way through the so called Hodge $\star$ operator. To give a very simplified definition for a metric g we locally have an orthogonal basis. The $\star$-operator associate a $k-$form to a $(n-k)$-form that is the "complement" in the basis. For example : $dx \to dy \wedge dz \wedge dt$ and $dx \wedge dt \to dy \wedge dz$[5].

Combined with the exterior derivative it give an operator $\partial = \star \circ d \circ \star$ that from a k-form gives an $(k-1)$-form.

[4] The orientation of the boundary parts is a bit tricky.

[5] With an appropriate sign depending on the signature of g.

3.2 Maxwell equations

Here are the Maxwell equations written with differential forms (we drop the constant ϵ_0, μ_0, c). Maxwell Theory is given by the wonderful table

$$
\begin{array}{ccccccccc}
\Lambda^0(\mathbb{R}^4) & \underset{\partial}{\overset{d}{\rightleftharpoons}} & \Lambda^1(\mathbb{R}^4) & \underset{\partial}{\overset{d}{\rightleftharpoons}} & \Lambda^2(\mathbb{R}^4) & \underset{\partial}{\overset{d}{\rightleftharpoons}} & \Lambda^3(\mathbb{R}^4) & \underset{\partial}{\overset{d}{\rightleftharpoons}} & \Lambda^4(\mathbb{R}^4) \\
 & & & & & & & & \\
f & \underset{(2)}{\overset{(1)}{\rightleftharpoons}} & (V, A) & \xrightarrow{(3)} & (E, B) & \xrightarrow{(4)} & 0 & & \\
 & & & & \star(E, B) & \xrightarrow[(5)]{} & (\rho, j) & \xrightarrow[(6)]{} & 0
\end{array}
$$

Notice that the dimension of $\Lambda^1(\mathbb{R}^4)$, $\Lambda^2(\mathbb{R}^4)$ and $\Lambda^3(\mathbb{R}^4)$ are 4, 6 and 4. We denote here

- $A^\mu = (V, A)$: the potential and potential vector (usually a $4-$vector). This is a $1-$form as it is linked to the phase of the wave function of the electron (as for example in the Aharonov Bohm effect).
- $F = (E, B)$: the electromagnetic field (usually already a antisymmetric $2-$tensor). This is a $2-$form, one would want to integrate the magnetic field on a surface for example.
- $J^\mu = (\rho, j)$: charge density and current (usually a 4-vector). This is a $3-$form, the density is integrate on a volume and the current on a surface times a time intervale.

A great advantage here that everything is at the "geometric" level. There is no choice of parametrisation of the space. No worry about the change of referencial.

Here every arrow of the diagramm is just the exterior derivative. Bellow are their usual meaning:

(1) This is a Gauge invariance, one can change

$$V \to V - \frac{\partial f}{\partial t} \quad \text{et} \quad A \to A + \mathrm{grad}(f)$$

without modifying electromagnetic field, indeed $(3) \circ (1) = 0$.

(2) A particular choice of Gauge called Lorentz's Gauge $(2) = 0$ that is

$$\frac{\partial V}{\partial t} + \mathrm{div}(A) = 0.$$

(3) Electromagnetic fields are expressed with the potential and potential vector:

$$E = -\frac{\partial A}{\partial t} - \mathrm{grad}(V) \quad \text{et} \quad B = \mathrm{curl}(A).$$

(4) Here are the Maxwell-Faraday et Maxwell-Thomson equation

$$\frac{\partial B}{\partial t} + \mathrm{curl}(E) = 0 \quad \text{et} \quad \mathrm{div}(B) = 0$$

This follows of course from $(4) \circ (3) = 0$.

(5) Here are now Maxwell-Gauss and Maxwell-Ampere equations

$$\mathrm{div}(E) = \rho \quad \text{et} \quad -\frac{\partial E}{\partial t} + \mathrm{curl}(B) = j$$

(6) This is the conservation law of the charge

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(j) = 0$$

Again that is $(6) \circ (5) = 0$.

We can finish by a small historical remark, in 1865 the fantastic idea of Maxwell were to notice that $(6) \circ (5) \neq 0$ with Ampere equation as stated at that time. He modified the equation adding the term $\frac{\partial E}{\partial t}$ to obtain a coherent theory. Therefore it is indeed a differential geometry approach that gave nowday's classical electromagnetic theory.

4 De Rham (Trivial) Cohomologie

We have seen in Section 2 that if a form α can be written as $\alpha = d\beta$ it satisfies $d\alpha = 0$. A natural question would be to asked whether the converse is always true. The problem gives rize to a whole domain of study called De Rham Cohomology which happens to be one of the most powerful tools to study and charaterize topological object in differential geometry and algebraic topology. Here we do not go deep into it but just mention one of its simplest result [God70].

Proposition 6. *(De Rham (trivial) Cohomology) In $\mathbb{R}^n$, for any form α that is not a constant $0-$form*

$$d\alpha = 0 \quad \Leftrightarrow \quad \text{there exists } \beta \text{ such that } \alpha = d\beta.$$

This proposition is not true for more complicated topological space and appends to be a very interesting mathematical question. But in $\mathbb{R}^3$ all this is very well known by any undergrad student :

- α is a 0-form : $\mathrm{grad}(\alpha) = 0$ iff there exists $c \in \mathbb{R}$ constant such that $\alpha = c$,
- α is a 1-form : $\mathrm{curl}(\alpha) = 0$ iff there exists a $0-$form β such that $\alpha = \mathrm{grad}(\beta)$,
- α is a 2-form : $\mathrm{div}(\alpha) = 0$ iff there exists a $1-$form β such that $\alpha = \mathrm{curl}(\beta)$,

- α is a 3-form : The equation $\mathrm{div}(\beta) = \alpha$ always has a solution.

In higher dimension, this also implies important properties.

Corollary 7. *If J is a conserved quantity, then there exists β such that $d\beta = J$.*

In particular in $\mathbb{R} \times \mathbb{R}^3$ for any (ρ, j) that satisfies the continuous equation there exists a field $\beta = (\mathcal{E}, \mathcal{B})$ for which Maxwell-Gauss and Maxwell-Ampere equations are valid

$$\mathrm{div}(\mathcal{E}) = \rho \quad \text{and} \quad -\frac{\partial \mathcal{E}}{\partial t} + \mathrm{curl}(\mathcal{B}) = j.$$

This is a purely mathematical construction. This field should be seen as an equivalent to the potential flow that describes an irrotational velocity field as a gradien in hydrodynamics.

Also such a β is not unique and one can made a choice of Gauge. In the case $d(\star\beta) = 0$ we have the following

Corollary 8. *If J is a conserved quantity, then there exists α such that $d\star d\alpha = J$.*

In a coordinate system it is the usual propagation equation with a source

$$\Box\alpha = J \tag{3}$$

We should stress that this is true for *any* conserved quantity with no more information about the physics of the system. However there are important examples where α appears in theorical model or is indeed a real physical quantity. For example :

- The electric charge is conserved. We have the electromagnetic field : (3) with $J = (\rho, j)$ the charge density and current and $\alpha = (V, A)$ potential and vector potential.

So we can ask the following.

Question 9. *What is the associated α for the conservation of energy, momentum, moment of inertia, weak charge,... ?*

5 Euler Lagrange Equations

Here we write the Lagrangian approach for a classical field theory on $\mathbb{R}^n$ using exterior forms.

5.1 Euler Lagrange Equations

As a most simple model we assume that the Lagrangian $\mathcal{L}(\alpha, d\alpha)$ is a n-form[6] that depends only on a $k-$form α and its exterior derivative $d\alpha$. The last

[6]Most commun conventions call the Lagrangian $\star\mathcal{L}$ which is then a 0-form (a scalar).

hypothesis is more restrictive than considering all the derivatives ($\frac{\partial \alpha}{\partial x_i}$) but is also reasonable if we believe that $d\alpha$ has more geometric or physical meaning.

We consider a small perturbation $\delta\alpha$ and write $\mathcal{L}_1 := \partial_1 \mathcal{L}(\alpha, d\alpha)$ that is an $(n-k)$-form and $\mathcal{L}_2 := \partial_2 \mathcal{L}(\alpha, d\alpha)$ that is a $(n-k-1)$-form such that at first order we have

$$\begin{aligned}\mathcal{L}(\alpha + \delta\alpha, d\alpha + d\delta\alpha) &\approx \mathcal{L}(\alpha, d\alpha) + \delta\alpha \wedge \mathcal{L}_1 + (d\delta\alpha) \wedge \mathcal{L}_2 \\ &= \mathcal{L}(\alpha, d\alpha) + \delta\alpha \wedge (\mathcal{L}_1 - (-1)^k d\mathcal{L}_2) + d(\delta\alpha \wedge \mathcal{L}_2)\end{aligned}$$

Maximising $\int_{\mathcal{V}} \mathcal{L}$ with fixed boundary conditions on $\partial\mathcal{V}$ gives the Euler-Lagrange Equation.

Definition 10. (Euler Lagrange Equation)

$$\mathcal{L}_1 - (-1)^k d\mathcal{L}_2 = 0$$

Remark that it directly implies that $d\mathcal{L}_1 = 0$. So in particular we have the following.

Corollary 11. *If α is a 1-form then $\mathcal{L}_1$ is a conserved quantity and $\mathcal{L}_2$ is an associated field*[7]*.*

A first nice example is of course the electromagnetic field with $\alpha = (V, A)$ and $J = (\rho, j)$ we have

$$\mathcal{L} = \alpha \wedge J - \frac{1}{2}(d\alpha \wedge \star d\alpha), \quad \mathcal{L}_1 = J \quad \text{and} \quad \mathcal{L}_2 = \star d\alpha$$

5.2 Noether Theorem

We now mention the famous Noether Theorem. Remark that Euler Lagrange equation implies that

$$\mathcal{L}(\alpha + \delta\alpha, d\alpha + d\delta\alpha) - \mathcal{L}(\alpha, d\alpha) \approx d(\delta\alpha \wedge \mathcal{L}_2)$$

We denote ξ an infinitesimal transformation $\alpha \to \alpha + \delta\alpha^\xi$ and $\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}^\xi$. If the system is invariant by this transformation $\delta\mathcal{L}^\xi = 0$ then $d(\delta\alpha^\xi \wedge \mathcal{L}_2) = 0$ so $\delta\alpha^\xi \wedge \mathcal{L}_2$ is a conserved quantity. We can state a more general theorem [Olv93].

Theorem 12. *(Noether Theorem) If $\delta\mathcal{L}^\xi = d(\delta\Lambda^\xi)$ then $\delta\alpha^\xi \wedge \mathcal{L}_2 - \delta\Lambda^\xi$ is a conserved quantity.*

A very nice application of Noether Theorem is of course the conservation of energy and momentum.

[7] As in Corollary 7

5.3 The stress-energy tensor

We consider translations of the system and more generaly the transport along a flow given by vector field X. Notice that we have $L_X\mathcal{L} = d(i_X\mathcal{L})$ (Cartan's magic formula) so we can apply Noether Theorem. We compute

$$L_X\mathcal{L} = \mathcal{L} + L_X\alpha \wedge \mathcal{L}_1 + (dL_X\alpha) \wedge \mathcal{L}_2 + L_X\mathcal{L}|_{\alpha,d\alpha}$$

and then obtain

$$d(L_X\alpha \wedge \mathcal{L}_2 - i_X\mathcal{L}) = -L_X\mathcal{L}|_{\alpha,d\alpha}.$$

For example in the case of the electromagnetism Lagrangian and $X = \partial_t$ for translation in time we obtain Poynting's theorem

$$\partial_t(\frac{1}{2}(|E|^2 + |B|^2) + \vec{A}.\vec{j}) + \mathrm{div}(E \times B) = \vec{A}.\partial_t\vec{j}.$$

For more general Lagrangian $\mathcal{L}$ we can define the following.

Definition 13. We call $L_X\alpha \wedge \mathcal{L}_2 - i_X\mathcal{L}$ the *stress-energy tensor* if X is a translation.

In a coordinate system and with the translations $X = \partial_\nu$, this is the usual formula for the stress-energy tensor defined from a Lagrangian

$$T_{\mu\nu} = \partial_\nu\alpha^\eta(\mathcal{L}_2)_{\mu\eta} - \eta_{\mu\nu}\mathcal{L}.$$

And we also state the conservation of energy and momentum in a general setting.

Corollary 14. *If $L_X\mathcal{L}|_{\alpha,d\alpha} = 0$, ie. the Lagrangian is invariant by the transformation induced by the vector field X, then $(L_X\alpha) \wedge \mathcal{L}_2 - i_X\mathcal{L}$ is a conserved quantity.*

5.4 gravitational waves ?

We finish by giving a partial answer for Question 9. One can find in a book on General Relativity this equation used to describe gravitational waves

$$\Box\tilde{h} = T$$

where $\tilde{h}$ is constructed with the perturbation of the metric g around the flat Minkowski metric and T is the 4×4 Stress-Energy tensor. Here one can think of each line of T as a 3-form which corresponds to the conservation of energy (first line) and the conservation of momentum (the three others lines) and then the line of $\tilde{h}$ play the role of the associated α in Corollary 8 and the associated β in Corollary 7 can be interpreted as the classical gravitational field.

Unfortunatly general relativity is much more complicated and the Conservation of energy/momentum is true only at first order when gravity is not too strong.

6 Summary table

$m^{-2}s^{-1}$	$dx \wedge dy \wedge dt$
gradient, curl, divergence	d
irrotational, divengence free, conserved quantity	$d\alpha = 0$
De Rham cohomology	$d\alpha = 0 \Rightarrow \alpha = d\beta$?
Maxwell equations	$dF = 0, \quad d \star F = 0$
Propagation wave with source	$d \star d\alpha = J$
Gauge invariance	$d(\alpha + d\beta) = d\alpha$
Euler Lagrange equation	$\mathcal{L}_1 - (-1)^k d\mathcal{L}_2 = 0$
Stress-energy tensor	$L_X\alpha \wedge \mathcal{L}_2 - i_X\mathcal{L}$

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