Test for the course “Martingale and brownian motion”

12 juin 2020

Length : 3 hours , The courses notes are not allowed.

1 Discrete Martingale.

Course question: Show that the maximum of many stopping time is a stopping time.

Problem 1. We consider a game with N players. Each player start with $X_0^{(i)}$ coins at the beginning. À each turn n, two players make a bet with one of theirs coin and play “head or tail”. The winner keeps the two coins. We denote the “head or tail” results $P_1, P_2, \cdots, P_n, \cdots$ that are iid random variable with $\mathbb{P}(P_1 = 1) = \mathbb{P}(P_2 = -1) = \frac{1}{2}$. We denote the natural filtration $\mathcal{F}_n = \sigma(P_1, \cdots, P_n)$. We denote the the players that bet at time n $(i_n, , j_n) \in [1, N]^2$. This is predictable process and for all n $X_n^{(i_n)} > 0$ et $X_n^{(j_n)} > 0$ (the two players have at least one coin). The game is over when all of the player exept one are ruined. This last one won the game.

1. Show that for all $1 \leq i \leq N$, $X^{(i)}$ is martingale.
2. Show that the game finish a.s. What is the final situation?
3. What is the probability that player i win the game.
4. Propose a predictable process $A_n^{(i)}$ such that $(X^{(i)})_n^2 - A_n^{(i)}$ is a martingale.
5. What is the expected number of bets “head or tail” that the player i will do during the game?
6. What is the expected length (total number of bets) of the game? For 10 players and $\sum_{i=1}^{10} X_0^{(i)} = 110$, for which initial configuration the expected length is maximum.
7. A cheater comes in the game. He starts with only one coin but for each bet “head or tail”, its winning probability is $\frac{2}{3}$ (and not $\frac{1}{2}$). For which probability will the cheater win the game?

2 Martingale continue

Course question: State and prove the integration by part for continuous martingales.

Problem 2. Let M_t a continuous martingale bounded in L^2. We denote $\langle M, M\rangle_t$ its quadratic variation and we have$\mathbb{E}(\langle M, M\rangle_1) = 1$. We define

$$S_n^{(N)} := \sum_{k=1}^{n} M_{\frac{k}{N}}(M_{\frac{k}{N}} - M_{\frac{k-1}{N}}).$$

1. Calculate $\mathbb{E}(S_N^{(N)})$. Is the process $(S_n^{(N)})_{n\in\mathbb{N}}$ a martingale?

2. Prove that $N \to \infty$, $S_N^{(N)}$ converge in probability. Give its limit.

3. Prove that $N \to \infty$, $\sum_{k=1}^{N}(M_{\frac{k}{N}} - M_{\frac{k-1}{N}})^3 \to 0$.

4. Solve the stochastic equation

$$N_t = 1 + \int_0^t N_s dB_s$$

with B_t a brownian motion.

Problem 3. We consider $\boldsymbol{B}_t = (B_t^{(1)}, B_t^{(2)})$ a brownian motion in $\mathbb{R}^2$ starting at $(0,0)$.($B_t^{(1)}, B_t^{(2)}$ are two independant brownian motion in $\mathbb{R}$).

1. Calculate $\mathbb{E}(\|\boldsymbol{B}_t\|^2)$ and prove that $\|\boldsymbol{B}_t\|^2 - \mathbb{E}(\|\boldsymbol{B}_t\|^2)$ is a martingale.

2. For $R > 0$ and $T := \inf\{t : \|\boldsymbol{B}_t\| = R\}$. Calculate $\mathbb{E}(T)$.

3. Let $x > 0$. In this question and the following $\boldsymbol{B}_t$ is a brownian motion starting at $(x, 0)$. We assume that a.s $B_t \neq 0$ for all $t > 0$. With Ito formula express $\|B_t\| = M_t + A_t$ with M_t a martingale and A_t a finite variation process.

4. Show that $\|B_t\|$ is a submartingale.

5. Show that M_t is a brownian motion.

6. Show that $\ln(\|B_t\|)$ is martingale. (One can use the polar coordinate Laplacien $\Delta f = \frac{1}{r}\frac{\partial}{\partial r}[r\frac{\partial}{\partial r}f] + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}$.)

7. For $0 < r < x < R$ $T_r = \inf\{t : \|\boldsymbol{B}_t\| = r\}$, $T_R = \inf\{t : \|\boldsymbol{B}_t\| = R\}$, Calculate $\mathbb{P}(T_r \leq T_R)$.

8. For $\boldsymbol{B}_t$ a brownian motion in dimension $d \geq 3$ starting at $(x, 0, \cdots, 0)$. Calculate $\mathbb{P}(T_r < \infty)$. (One can propose an harmonic function on $\mathbb{R}^d/\{0\}$).