Extremal eigenvalues of critical Erdős-Rényi graphs

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Abstract

We complete the analysis of the extremal eigenvalues of the the adjacency matrix A of the Erdős-Rényi graph $G(N, d/N)$ in the critical regime $d \asymp \log N$ of the transition uncovered in [2,3], where the regimes $d \gg \log N$ and $d \ll \log N$ were studied. We establish a one-to-one correspondence between vertices of degree at least $2d$ and nontrivial (excluding the trivial top eigenvalue) eigenvalues of $A/\sqrt{d}$ outside of the asymptotic bulk $[-2, 2]$. This correspondence implies that the transition characterized by the appearance of the eigenvalues outside of the asymptotic bulk takes place at the critical value $d = d_* = \frac{1}{\log 4 - 1} \log N$. For $d < d_*$ we obtain rigidity bounds on the locations of all eigenvalues outside the interval $[-2, 2]$, and for $d > d_*$ we show that no such eigenvalues exist. All of our estimates are quantitative with polynomial error probabilities.

Our proof is based on a tridiagonal representation of the adjacency matrix and on a detailed analysis of the geometry of the neighbourhood of the large degree vertices. An important ingredient in our estimates is a matrix inequality obtained via the associated nonbacktracking matrix and an Ihara-Bass formula [2]. Our argument also applies to sparse Wigner matrices, defined as the Hadamard product of A with a Wigner matrix, in which case the role of the degrees is replaced by the squares of the ℓ^2-norms of the rows.

1. Introduction

This paper is about the extremal eigenvalues of sparse random matrices, such as the adjacency matrix of the Erdős-Rényi graph. In spectral graph theory, obtaining precise bounds on the locations of the extreme eigenvalues, in particular on the spectral gap, is of fundamental importance and has attracted much attention in the past thirty years. See for instance [1, 5, 10] for reviews.

The Erdős-Rényi graph $G = G(N, d/N)$ is the simplest model of a random graph, where each edge of the complete graph on N vertices is kept independently with probability d/N, with $0 < d < N$. Its adjacency matrix A is the canonical example of a sparse random matrix, and its spectrum has been extensively studied in the random matrix theory literature. In the regime $d \equiv d_N \to \infty$ as $N \to \infty$, the empirical eigenvalue measure of $A/\sqrt{d}$ converges to the semicircle law supported on $[-2, 2]$ [17, 20].

The behaviour of the extremal eigenvalues is more subtle, and has been investigated in several recent works [2,3,6–9,11–14,19]. In particular, in [12] it is shown that the largest eigenvalue $\lambda_1(A)$ of A is asymptotically equivalent to the maximum of d and the square root of the largest degree of G. A more difficult question is that of the other eigenvalues, $\lambda_2(A), \dots, \lambda_N(A)$, which determine in particular the gap $\lambda_1(A) - \lambda_2(A)$ between the largest and second-largest eigenvalues. By a standard eigenvalue interlacing argument, the analysis of the extremal eigenvalues $\lambda_2(A), \dots, \lambda_N(A)$ of A is equivalent to the analysis of the eigenvalues $\lambda_1(\underline{A}), \dots, \lambda_N(\underline{A})$ of the centred adjacency matrix $\underline{A} := A - \mathbb{E}A$.

An important motivation for the present work is a transition in the behaviour of the extremal eigenvalues of $\underline{A}$ observed in [2,3]. In [3] it is shown that in the regime $d \gg \log N$ the extremal eigenvalues $\lambda_1(\underline{A}/\sqrt{d})$ and $\lambda_N(\underline{A}/\sqrt{d})$ converge with high probability to the edges ± 2 of the semicircle law's support. Conversely, in [2] it is shown that in the regime $d \ll \log N$, the extremal eigenvalues $\lambda_1(\underline{A}/\sqrt{d})$ and $\lambda_N(\underline{A}/\sqrt{d})$ are asymptotically of order $\pm\sqrt{\eta/\log \eta}$ with $\eta := \frac{\log N}{d}$, placing them far outside of the interval $[-2,2]$.

Based on the two different behaviours observed in [2,3], we therefore expect a transition in the behaviour of the extremal eigenvalues at the *critical* density $d \asymp \log N$, where the extremal eigenvalues leave the support of the semicircle law.

In this paper we give a detailed analysis of this transition around the critical scale $d \asymp \log N$, which was left open by the works [2,3], by deriving quantitative high-probability bounds on the locations of all eigenvalues of $A/\sqrt{d}$ and $\underline{A}/\sqrt{d}$ that lie outside the interval $[-2,2]$. Our analysis covers also the neighbouring sub- and supercritical regimes, $d \ll \log N$ and $d \gg \log N$, and in particular provides a complete pictures of the transition between these two regimes. Our approach also works for sparse Wigner matrices of the form $X = (X_{xy})_{x,y=1}^N$, where $X_{xy} = W_{xy}A_{xy}$ and $(W_{xy} : x \leqslant y)$ are uniformly bounded independent random variables with zero expectation and unit variance.

We remark that the critical scale $d \asymp \log N$ is the same as the well-known connectivity threshold for the Erdős-Rényi, which happens precisely at the value $d = \log N$. In contrast, although the transition in the locations of the extremal eigenvalues of $\underline{A}$ happens on the same scale $d \asymp \log N$, it happens at a different numerical value, $d = \frac{1}{\log 4 - 1} \log N \approx 2.59 \log N$.

The mechanism underlying the emergence of eigenvalues outside the support of the semicircle distribution for sufficiently sparse matrices is the appearance of vertices of large degree. This was already observed and exploited in [2] in the subcritical regime $d \ll \log N$. The intuition is that for sufficiently small d, the concentration of the degrees of the vertices around their mean d fails, and we observe a number of vertices whose degree is much larger than d. This mechanism is also at the heart of our analysis. In fact, our main result is a high-probability correspondence between vertices of large degree and extremal eigenvalues. Roughly, we show that the following holds with probability at least $1 - N^{-\nu}$ for any fixed $\nu > 0$.

(i) Every vertex x with degree D_x larger than $(2 + o(1))d$ gives rise to exactly one eigenvalue of $\underline{A}/\sqrt{d}$ in $[2 + o(1), \infty)$ and one in $(-\infty, -2 - o(1)]$. These eigenvalues are located near $\pm\Lambda(D_x/d)$ respectively, where $\Lambda(t) := \frac{t}{\sqrt{t-1}}$. The error is bounded by an inverse power of d.

(ii) There are no other eigenvalues in $(-\infty, -2 - o(1)] \cup [2 + o(1), \infty)$.

Using standard results on the degree distribution of the Erdős-Rényi graph (for the reader's convenience we review the necessary results in Appendix D), we can then easily conclude rigidity estimates for all eigenvalues of $\underline{A}/\sqrt{d}$ and $A/\sqrt{d}$ in the region $\mathbb{R} \setminus [-2 - o(1), 2 + o(1)]$.

Our proof is based on the tridiagonal representation [18] of the matrix $\underline{A}$ around some vertex x. Thus, for any vertex $x \in [N]$ we consider the unit vector $\mathbf{1}_x$ supported at x and rewrite $\underline{A}$ in the basis obtained by orthonormalizing the vectors $\mathbf{1}_x, \underline{A}\mathbf{1}_x, \underline{A}^2\mathbf{1}_x, \dots$. The resulting matrix M is tridiagonal and its spectrum coincides with that of $\underline{A}$. Denoting by $S_i(x)$ the sphere of radius i around x, the key intuition behind our proof is that even though $D_x = |S_1(x)|$ does not concentrate in the critical and subcritical regimes, the *quotients* $|S_{i+1}|/|S_i|$, $i \geqslant 1$, do. Moreover, we note that balls of sufficiently small radius have only a bounded number of cycles with high probability, and can therefore be approximated by trees after a removal of a bounded number of edges. Thus, we expect the tridiagonal matrix M to be close to that of a tree whose root x has D_x children and all other vertices d children (see (3.3) and Figure 3.1 below). The spectrum of this latter matrix may be analysed using transfer matrix methods. We remark that this approximation requires precise information about the geometry of the neighbourhoods of vertices, and is only correct for vertices of large enough degree.

In practice, we proceed as follows. For clarity, let us focus only on the positive eigenvalues. Our proof then consists of two major steps: deriving lower and upper bounds on the extremal eigenvalues of $\underline{A}$. For the lower bounds, we construct approximate eigenvectors $\mathbf{v}^{(x)}$ of $\underline{A}$ around vertices x of high degree, whose definition is motivated by the fact that $\mathbf{v}^{(x)}$ would be an exact eigenvector if the approximation by a regular tree sketched above were exact. In addition to showing that these $\mathbf{v}^{(x)}$ are indeed approximate eigenvectors with a quantitatively controlled error bound, we need to show that all of the associated eigenvalues in $[2+o(1),\infty)$ are distinct. We do this by a careful pruning of the graph, with the property that all balls (in the pruned graph) of suitable radii around the vertices $\mathcal{V}_2 := \{x \in [N] : D_x \geqslant 2d\}$ are disjoint, and that the degrees of the difference between the original and pruned graphs are not too large. Since $\mathbf{v}^{(x)}$ is supported in a sufficiently small ball around x, this will imply that the family $(\mathbf{v}^{(x)})_{x\in\mathcal{V}_2}$ is orthogonal, and hence the associated eigenvalues of $\underline{A}/\sqrt{d}$ are distinct.

For the matching upper bounds on the extremal eigenvalues, a fundamental input is an Ihara-Bass type formula and a bound on the spectral radius of the *nonbacktracking matrix* associated with $\underline{A}$ derived in [3]. This argument allows us to completely bypass typically very complicated combinatorial arguments needed in the trace method for estimating matrix norms. Thanks to the Ihara-Bass formula, the trace method is performed only on the level of the nonbacktracking matrix; this was already performed in [3] using a trace method that was very simple thanks to the nonbacktracking property. In particular, the lack of concentration of the degrees, which has a crucial impact on the extremal eigenvalues of $\underline{A}$, has no impact on the extremal eigenvalues (in absolute value) of the nonbacktracking matrix of $\underline{A}$. The outcome of this observation is the matrix inequality $\underline{A}/\sqrt{d} \leqslant I_N + D + o(1)$, where D is the diagonal matrix with entries D_x/d. We apply this inequality to estimate the norm of the matrix $\underline{A}/\sqrt{d}$ restricted to vertices with degrees at most $2d$, and show that it is bounded by $2+o(1)$. To that end we need to derive, for the maximal eigenvector of the restricted matrix, a delocalization bound at vertices with degree at least $(1+o(1))d$. This delocalization bound is derived using a careful analysis of the tridiagonal matrix associated with the restricted adjacency matrix. In fact, all of this analysis has to be done with the pruned adjacency matrix described above in order to obtain simultaneous upper bounds on all eigenvalues down to $2+o(1)$.

The argument sketched above can also be easily applied to the sparse Wigner matrices X described above, essentially by replacing the degree D_x of a vertex by the square ℓ^2-norm of the x-th row of X. We refer to Section 3.2 below for a more detailed summary of the proof.

Our method is rather general and in particular it is not tied to the homogeneity of the Erdős-Rényi graph. We therefore expect it to be applicable to many other sparse random matrix models, such as inhomogeneous Erdős-Rényi graphs and stochastic block models.

We remark that related results appeared in the independent work [16] while we were finalizing the current manuscript. In [16], the authors show that, for any fixed $k \in \mathbb{N}$, the largest / smallest k eigenvalues of $\underline{A}$ are with high probability equal to $\pm(1+o(1))\sqrt{d}\Lambda(D_{\max}/d \vee 2)$. Their argument also works for sparse Wigner matrices X described above. In particular, the precise location $d = \frac{1}{\log 4 - 1}\log N$ for the transition in the behaviour of the top extremal eigenvalues of the Erdős-Rényi graph was also established in [16]. The proof of [16] differs substantially from ours; it relies on suitably chosen trial vectors and an intricate trace method argument controlled using cleverly constructed data structures.

We conclude the introduction with a brief outline of the paper. In Section 2 we state our results. The rest of the paper is devoted to the proofs. In Section 3, we introduce notations used throughout the paper and give a more detailed summary of the proof. In Section 4, we show that a vertex of degree greater than $2d$ induces two approximate eigenvectors for the adjacency matrix. The subsequent Section 5 is devoted to a quadratic form bound on the adjacency matrix in terms of the degree matrix. Lower and upper bounds on large eigenvalues of the adjacency matrix are established in Section 6 and Section 7, respectively. In the short Section 8 we put everything together and conclude our main results for the Erdős-Rényi graph. In the Section 9, we explain the minor changes required to handle sparse Wigner matrices. In the appendices, we collect some basic results on tridiagonal matrices and the degree distribution of Erdős-Rényi graphs.

Convention. We regard N as the fundamental large parameter. All quantities that are not explicitly *fixed* may depend on N; we almost always omit the argument N from our notation.

2. Results

Let $A \in \{0,1\}^{N\times N}$ be the adjacency matrix of the homogeneous Erdős-Rényi graph with vertex set $[N] := \{1,\dots,N\}$ and edge probability d/N. That is, $A = A^*$, $A_{xx} = 0$ for all $x \in [N]$, and $(A_{xy} \colon x < y)$ are independent Bernoulli(d/N) random variables. Throughout this paper, N is a large parameter and $d \equiv d_N$ depends on N. For each $x \in [N]$, we define the *normalized degree* α_x of x through

$$\alpha_x := \frac{1}{d}\sum_{y\in[N]} A_{xy}. \tag{2.1}$$

We also consider the centered adjacency matrix $\underline{A} := A - \mathbb{E}A$. For any Hermitian matrix $M = M^* \in \mathbb{R}^{N\times N}$, we denote by $\lambda_1(M) \geqslant \lambda_2(M) \geqslant \ldots \geqslant \lambda_N(M)$ its eigenvalues.

For $t > 1$ we define

$$\Lambda(t) := \frac{t}{\sqrt{t-1}}\,. \tag{2.2}$$

We denote by $\sigma\colon [N] \to [N]$ a random permutation such that

$$\alpha_{\sigma(1)} \geqslant \alpha_{\sigma(2)} \geqslant \cdots \geqslant \alpha_{\sigma(N)}. \tag{2.3}$$

We can now state our main result.

Theorem 2.1. *Fix $0 < \kappa < 1/2$. Abbreviate $a := 1/2 - \kappa$ and suppose that $\kappa \leqslant \theta < 5/2$. Suppose that $(\log N)^{4/(5-2\theta)} \leqslant d \leqslant N^{2/13}$. Define the random index*

$$L := \max\{l \geqslant 1 : \alpha_{\sigma(l)} \geqslant 2 + (\log d)^{a}\}$$

with the convention that $L = 0$ if $\alpha_{\sigma(1)} < 2 + (\log d)^{-a}$. Then there is a universal constant c such that for any $\nu > 0$ there is a constant $\mathcal{C} \equiv \mathcal{C}_{\nu,\kappa}$ such that the following holds with probability at least $1 - \mathcal{C}N^{-\nu}$.

(i) For $1 \leqslant l \leqslant L$ we have

$$\big|\lambda_l(\underline{A}) - \sqrt{d}\Lambda(\alpha_{\sigma(l)})\big| + \big|\lambda_{N+1-l}(\underline{A}) + \sqrt{d}\Lambda(\alpha_{\sigma(l)})\big| \leqslant \mathcal{C}\sqrt{d}\left(d^{-c(\Lambda(\alpha_{\sigma(l)})-2)} + d^{-\theta/2}\right).$$

(ii) For $l = L+1$ we have

$$\max\{\lambda_l(\underline{A}), -\lambda_{N+1-l}(\underline{A})\} \leqslant 2\sqrt{d} + \mathcal{C}\sqrt{d}(\log d)^{-2a}\,.$$

Remark 2.2. In the supercritical regime $d \gg \log N$, Theorem 2.1 is established in [2], and in the subcritical regime $d \ll \log N$ it is established in [3], in both cases with quantitative error bounds (which however differ from those of Theorem 2.1). Hence, in Theorem 2.1 it would be sufficient to assume that $d \asymp \log N$. We allow a larger range so as to obtain a simple statement that covers all three regimes, showcasing the full behaviour through the transition at criticality.

As a consequence of Theorem 2.1, for any $\nu > 0$ there is a constant $\mathcal{C} \equiv \mathcal{C}_\nu > 0$ such that, with probability at least $1 - \mathcal{C}N^{-\nu}$,

$$\|\underline{A}\| = \sqrt{d}(\Lambda(\alpha_{\sigma(1)}) \vee 2)(1 + o(1))\,. \tag{2.4}$$

Another easy consequence is the corresponding statement for the non-centred adjacency matrix A, which follows by eigenvalue interlacing.

Corollary 2.3. *Under the same conditions and notations as in Theorem 2.1, the following holds with probability at least* $1 - \mathcal{C}N^{-\nu}$*.*

(i) For $1 \leqslant l \leqslant L$ *we have*

$$\begin{aligned}|\lambda_{l+1}(A) - \sqrt{d}\Lambda(\alpha_{\sigma(l)})| + |\lambda_{N+1-l}(A) + \sqrt{d}\Lambda(\alpha_{\sigma(l)})| \\ \leqslant \mathcal{C}\sqrt{d}\left(d^{-c(\Lambda(\alpha_{\sigma(l)})-2)} + d^{-\theta/2} + (\alpha_{\sigma(l)} - \alpha_{\sigma(l+1)})\right).\end{aligned}$$

(ii) For $l = L + 1$ *we have*

$$\max\{\lambda_{l+1}(A), -\lambda_{N+1-l}(A)\} \leqslant 2\sqrt{d} + \mathcal{C}\sqrt{d}(\log d)^{-2a}\,.$$

Note that the additional error term $(\alpha_{\sigma(l)} - \alpha_{\sigma(l+1)})$ is of order $1/d$ with high probability (see Proposition D.1 below). It is well known that the largest eigenvalue $\lambda_1(A)$ is an outlier far outside the bulk spectrum; in fact a trivial perturbation argument using (2.4) implies that $|\lambda_1(A) - d| \leqslant \sqrt{d}(\Lambda(\alpha_{\sigma(1)}) \vee 2)(1 + o(1))$ with probability at least $1 - \mathcal{C}N^{-\nu}$, where $\nu > 0$ and $\mathcal{C} \equiv \mathcal{C}_\nu$.

Theorem 2.1 (and its non-centred counterpart) can be combined with a standard analysis of the distribution of the degree sequence $D_{\sigma(1)}, D_{\sigma(2)}, \dots$ of the Erdős-Rényi graph. For the convenience of the reader, in Appendix D we collect some basic results about the degree distribution. As an illustration, we state such an application for the extremal eigenvalues of A.

For its statement, we need the following facts from Appendix D. For any $d > 0$ and $1 \leqslant l \leqslant \frac{N}{C\sqrt{d}}$, the equation

$$d(\beta \log \beta - \beta + 1) + \frac{1}{2}\log(2\pi\beta d) = \log(N/l)$$

has a unique solution $\beta_l(d)$. (Here C is a universal constant.) The interpretation of $\beta_l(d)$ is the typical value of the normalized degree $\alpha_{\sigma(l)}$. There is a typical normalized degree greater than or equal to 2 if and only if $\beta_1(d) \geqslant 2$. Thus, we introduce the critical value d_* as the unique solution of $\beta_1(d_*) = 2$. It is easy to see that

$$d_* = c_* \log N - \frac{c_*}{2}\log\log N + O(1)\,, \qquad c_* := \frac{1}{\log 4 - 1}\,. \tag{2.5}$$

Then Corollary 2.3 and Proposition D.1 imply the following result.

Corollary 2.4. *Under the same conditions and notations as in Theorem 2.1, the following holds. Define the deterministic index*

$$L(d) := \max\{l \geqslant 1 : \beta_l(d) \geqslant 2 + (\log d)^a\}$$

(i) For $1 \leqslant l \leqslant L(d)$ we have with probability $1 - o(1)$

$$\left|\lambda_{l+1}(A) - \sqrt{d}\Lambda(\beta_l(d))\right| + \left|\lambda_{N+1-l}(A) + \sqrt{d}\Lambda(\beta_l(d))\right| \leqslant C\sqrt{d}\left(d^{-c(\Lambda(\beta_l(d))-2)} + d^{-\theta/2}\right).$$

(ii) For $l = L(d) + 1$ we have with probability $1 - o(1)$

$$\max\{\lambda_{l+1}(A), -\lambda_{N+1-l}(A)\} \leqslant 2\sqrt{d} + C\sqrt{d}(\log d)^{-2a}.$$

(Here $C \equiv C_\kappa$ is a constant depending on κ.)

An analogous result holds for the matrix $\underline{A}$, whose details we omit.

Our final result is a version of our results for sparse Wigner matrices. Let $A = A_{xy}$ be as above and $W = (W_{xy})$ be an independent Wigner matrix with bounded entries. That is, W is Hermitian and its upper triangular entries $(W_{xy} : x \leqslant y)$ are independent random variables with mean zero and variance one, and $|W_{xy}| \leqslant C$ almost surely for some constant C. Then we define the sparse Wigner matrix $X = (X_{xy})$ as the Hadamard product $X_{xy} := A_{xy}W_{xy}$.

Theorem 2.5. *Theorem 2.1 holds also for the eigenvalues $\lambda_l(X)$ of a sparse Wigner matrix instead of $\lambda_l(\underline{A})$, provided that the normalized degree α_x is replaced by*

$$\alpha_x = \frac{1}{d}\sum_{y\in[N]} X_{xy}^2. \tag{2.6}$$

3. Notations and main ideas of the proof

In this section, we collect a few notations and tools used throughout the present work. Moreover, in Subsection 3.2, we explain the main ideas of the proof of Theorem 2.1. The reader can jump to Subsection 3.2 immediately and consult Subsection 3.1 for unknown notations.

3.1. Notations. We denote the positive integers by $\mathbb{N} = \{1, 2, 3, \ldots\}$ and define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We set $[n] := \{1, \ldots, n\}$ for any $n \in \mathbb{N}$ and $[\![r]\!] := \{0, \ldots, r\}$ for any $r \in \mathbb{N}_0$. We write $|X|$ for the cardinality of the finite set X. We use $\mathbb{1}_\Omega$ as symbol for the indicator function of the event Ω. Universal constants or estimates involving a universal constant are denoted by C and $O(\,\cdot\,)$, respectively.

Notations related to vectors and matrices. Vectors in $\mathbb{R}^N$ are denoted by bold faced small Latin letters like $\mathbf{u}$, $\mathbf{v}$ or $\mathbf{w}$ and their Euclidean norms by $\|\mathbf{u}\|$, $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$, respectively. For a matrix $M \in \mathbb{R}^{N\times N}$, $\|M\|$ is its operator norm induced by the Euclidean norm on $\mathbb{R}^N$.

Let $M \in \mathbb{R}^{N\times N}$ be a matrix and $V \subset [N]$. We define the matrix $M_V \in \mathbb{R}^{|V|\times|V|}$ and the family $M_{(V)}$ through

$$M_V := (M_{ij})_{i,j\in V}, \qquad M_{(V)} := (M_{ij})_{i\in V \text{ or } j\in V}.$$

If $V = \{x\}$ for some $x \in [N]$ then we also write $M_{(x)}$ instead of $M_{(\{x\})}$.

For a Hermitian matrix $M \in \mathbb{R}^{N\times N}$, its eigenvalues are denoted by

$$\lambda_1(M) \geqslant \lambda_2(M) \geqslant \cdots \geqslant \lambda_N(M).$$

Moreover, for Hermitian matrices $R, T \in \mathbb{R}^{N\times N}$ we write $R \geqslant T$ if

$$\langle \mathbf{w}, R\mathbf{w}\rangle \geqslant \langle \mathbf{w}, T\mathbf{w}\rangle$$

for all $\mathbf{w} \in \mathbb{R}^N$. We remark that this is equivalent to $\lambda_N(R-T) \geqslant 0$.

For any $x \in [N]$, we define the vector $\mathbf{1}_x := (\delta_{xy})_{y\in[N]} \in \mathbb{R}^N$ which is one in the x-component and vanishes otherwise. To any subset $S \subset [N]$, we associate the vector $\mathbf{1}_S \in \mathbb{R}^N$ given by $\mathbf{1}_S := \sum_{x\in S} \mathbf{1}_x$. Note that $\mathbf{1}_{\{x\}} = \mathbf{1}_x$. We also introduce the normalized vector $\mathbf{e} := N^{-1/2}\mathbf{1}_{[N]}$. If $V \subset [N]$ and $\mathbf{w} = (w_i)_{i\in[N]} \in \mathbb{R}^N$ then $\mathbf{w}|_V$ denotes the vector in $\mathbb{R}^N$ with components $\langle \mathbf{1}_y , \mathbf{w}|_V \rangle := \langle \mathbf{1}_y , \mathbf{w}\rangle$ for all $y \in V$ and $\langle \mathbf{1}_y , \mathbf{w}|_V \rangle = 0$ for all $y \in [N] \setminus V$.

Notations related to graphs. In the entire paper, we consider finite graphs exclusively. Let H and G be two graphs. We write $H \subset G$ if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. If $H \subset G$ then we denote by $G\backslash H$ the graph on $V(G)$ with edge set $E(G)\backslash E(H)$. To each graph $G = (V(G), E(G))$ we assign its adjacency matrix $\mathrm{Adj}(G)$. If G is a graph on $[N]$ then, for any $V \subset [N]$, we denote by $G|_V$ the subgraph induced by G on the vertex set V. If A is the adjacency matrix of G then $A_V = \mathrm{Adj}(G|_V)$ is the adjacency matrix of $G|_V$.

For simplicity, we specialize to the vertex set $[N]$ in the following definitions. Let H be a graph with vertex set $[N]$ and $M = \mathrm{Adj}(H)$ be its adjacency matrix. Vertices in $[N]$ are usually labelled by x, y, z. The degree of the vertex x is $D_x^H := \sum_{y\in[N]} M_{xy}$. With respect to H, the graph distance of two vertices $x, y \in [N]$ is denoted by

$$d^H(x,y) := \min\{k \in \mathbb{N}_0 : (M^k)_{xy} \neq 0\}\,.$$

For $i \in \mathbb{N}_0$, we introduce the i-sphere $S_i^H(x)$ and the i-ball $B_i^H(x)$ around x defined through

$$S_i^H(x) = \{y \in [N] \colon d^H(x,y) = i\}, \qquad B_i^H(x) = \{y \in [N] \colon d^H(x,y) \leqslant i\}.$$

For the remainder of this work, G will be an Erdős-Rényi graph with vertex set $[N]$ and edge probability d/N, where N is a large parameter and $d \equiv d_N$ is a function of N. Moreover, $A = \mathrm{Adj}(G) = (A_{xy})_{x,y\in[N]} \in \{0,1\}^{N\times N}$ will always denote the adjacency matrix of G. In this situation, we write D_x, $d(x,y)$, $S_i(x)$ and $B_i(x)$ instead of D_x^G, $d^G(x,y)$, $S_i^G(x)$ and $B_i^G(x)$, respectively. Note the relation $\alpha_x d = D_x$ between the normalized degree α_x defined in (2.1) and the degree D_x.

Probabilistic notations and tools. We now introduce a notion of very high probability event as well as a notation for bounds which hold with very high probability. Both will be used extensively throughout the present work.

Definition 3.1 (Very high probability)**.** (i) Let $\Xi \equiv \Xi_{N,\nu}$ be a family of events parametrized by $N \in \mathbb{N}$ and $\nu > 0$. We say that Ξ *holds with very high probability* if for every $\nu > 0$ there exists $\mathcal{C}_\nu$ such that

$$\mathbb{P}(\Xi_{N,\nu}) \geqslant 1 - \mathcal{C}_\nu N^{-\nu}$$

for all $N \in \mathbb{N}$.

(ii) For a σ-algebra $\mathcal{F}_N$ and an event $E_N \in \mathcal{F}_N$, we extend the definition (i) to Ξ *holds with very high probability on E conditioned on $\mathcal{F}$* if for all $\nu > 0$ there exists $\mathcal{C}_\nu$ such that

$$\mathbb{P}(\Xi_{N,\nu} \,|\, \mathcal{F}_N) \geqslant 1 - \mathcal{C}_\nu N^{-\nu}$$

almost surely on E_N, for all $N \in \mathbb{N}$.

We remark that the notion of very high probability survives a union bound involving $N^{O(1)}$ events. We shall tacitly use this fact throughout the paper.

Convention 3.2 (Estimates with very high probability)**.** In statements that hold with very high probability, we use the symbol $\mathcal{C} \equiv \mathcal{C}_\nu$ to denote a generic positive constant depending on ν such that the statement holds with probability at least $1 - c_\nu N^{-\nu}$ provided $\mathcal{C}_\nu$ and c_ν are chosen large enough.

We now illustrate the previous convention by explaining in detail the meaning of $|X| \leqslant \mathcal{C}Y$ with very high probability. Such estimates often appear throughout the paper. The bound $|X| \leqslant \mathcal{C}Y$ *with very high probability* means that, for each $\nu > 0$, there are constants $\mathcal{C}_\nu > 0$ and $c_\nu > 0$, depending on ν, such that

$$\mathbb{P}\big(|X| \leqslant \mathcal{C}_\nu Y\big) \geqslant 1 - c_\nu N^{-\nu}$$

for all $N \in \mathbb{N}$. Here, X and Y are allowed to depend on N.

We also write $X = \mathcal{O}(Y)$ to mean $|X| \leqslant \mathcal{C}Y$.

Throughout the following we use the function

$$h(\alpha) := (1+\alpha)\log(1+\alpha) - \alpha$$

for $\alpha \geqslant 0$.

To illustrate Definition 3.1 and Convention 3.2, we record the following lemma that we shall need throughout the paper.

Lemma 3.3 (Upper bound on the degree)**.** *For any $x \in [N]$ we have with very high probability*

$$D_x \leqslant \Delta \leqslant \mathcal{C}(d + \log N),$$

where $\Delta \equiv \Delta(d, N, \mathcal{C})$ is defined by

$$\Delta := \begin{cases} d + \mathcal{C}\sqrt{d \log N} & \text{if } d \geqslant \frac{1}{2}\log N \\ \mathcal{C}\frac{\log N}{\log\log N - \log d} & \text{if } d \leqslant \frac{1}{2}\log N. \end{cases} \tag{3.1}$$

Proof. From Bennett's inequality we obtain

$$\mathbb{P}(D_x \geqslant d + \alpha d) \leqslant \mathrm{e}^{-dh(\alpha)}\,.$$

The claim now follows from an elementary analysis of the right-hand side, by requiring that it be bounded by $N^{-\nu}$. ☐

3.2. Main ideas of the proof. In this subsection, we explain the main ideas of the proof of Theorem 2.1. Let G be an Erdős-Rényi graph with vertex set $[N]$ and edge probability d/N and let A be its adjacency matrix. In the actual proof, all arguments will be applied to $\underline{A} = A - \mathbb{E}A$. However, in this sketch, we explain certain ideas on the level of A for the sake of clarity. In each case, a simple adjustment yields the argument for $\underline{A}$ instead of A.

If $d \ll \log N$, then A has many eigenvalues of modulus larger than $2\sqrt{d}$ and they are related to vertices of large degree [2]. On the other hand, if $d \gg \log N$ then there are no eigenvalues whose modulus is larger than $2\sqrt{d}$ [3].

In order to understand the relationship between large eigenvalues and vertices of large degree it is very insightful to analyse the structure of G in the neighbourhood of a vertex $x \in [N]$ of large normalized degree α_x. (In the following, we explain the arguments for large eigenvalues only. Dealing with small eigenvalues requires straightforward modifications.) If α_x is sufficiently large then there is $r_x \in \mathbb{N}$, depending on α_x, such that G has with very high probability the following properties.

(a) For each $1 \leqslant i \leqslant r_x$, the ratio $|S_{i+1}(x)|/|S_i(x)|$ concentrates around d (Lemma 4.4 below).

(b) The subgraph $G|_{B_{r_x}(x)}$ is a tree up to a bounded number of edges (Lemma 4.5 below).

(c) The radius r_x tends to infinity with N (cf. (4.1) below).

Owing to the properties (a) and (b) of the local geometry of G around a vertex x of large degree, it is natural to study the spectral properties of the adjacency matrix of the following idealized graph $\mathcal{T}$ on $[N]$. We suppose that in the ball $B^{\mathcal{T}}_{r_x+1}(x)$ the graph $\mathcal{T}$ is a tree where the root vertex x has $d\alpha_x$ children and the vertices in $B^{\mathcal{T}}_{r_x}(x) \setminus \{x\}$ have d children. See Figure 3.1 for an illustration.

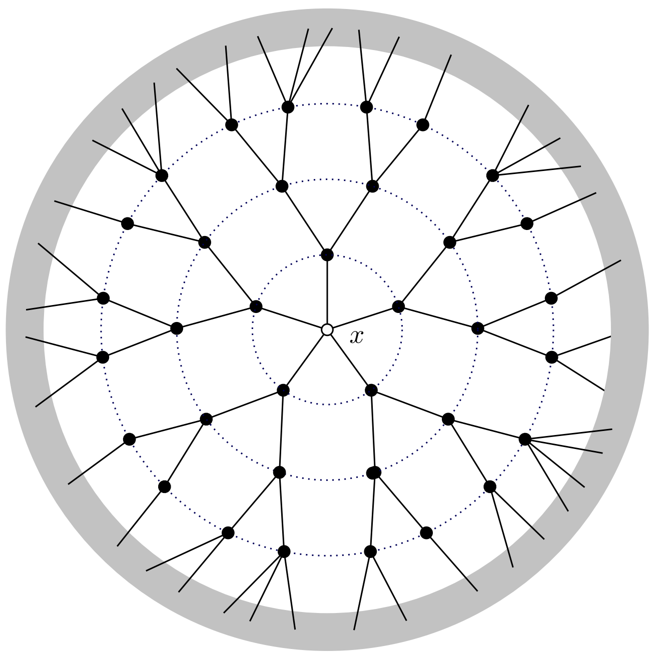

Figure 3.1. The regular tree graph $\mathcal{T}$ with $r_x = 2$, $d = 2$, and $D_x = d\alpha_x = 5$. We only draw vertices in the ball $B^{\mathcal{T}}_{r_x+1}(x)$, while the remaining vertices in $[N] \setminus B^{\mathcal{T}}_{r_x+1}(x)$ are in the grey area.

The adjacency matrix associated with $\mathcal{T}$ is denoted by $A_{\mathcal{T}}$. The following standard construction [18] yields a convenient approach to the spectral analysis of $A_{\mathcal{T}}$. Let $\mathbf{s}_0, \dots, \mathbf{s}_{r_x}$ be the Gram-Schmidt orthonormalisation of $\mathbf{1}_x, (A_{\mathcal{T}})\mathbf{1}_x, \dots, (A_{\mathcal{T}})^{r_x}\mathbf{1}_x$, where $\mathbf{1}_x = (\delta_{xy})_{y\in[N]} \in \mathbb{R}^N$. Let $\mathbf{s}_{r_x+1}, \dots, \mathbf{s}_{N-1}$ be any completion of $\mathbf{s}_0, \dots, \mathbf{s}_{r_x}$ to an orthonormal basis of $\mathbb{R}^N$. We denote by $M_{\mathcal{T}}$ the matrix representation of $A_{\mathcal{T}}$ in this basis, i.e.,

$$M_{\mathcal{T}} = S^* A_{\mathcal{T}} S, \qquad S := (\mathbf{s}_0, \dots, \mathbf{s}_{N-1}) \in \mathbb{R}^{N\times N}. \tag{3.2}$$

Note that $A_{\mathcal{T}}$ and $M_{\mathcal{T}}$ have the same spectrum. The upper-left $(r_x+1)\times(r_x+1)$ block $(M_{\mathcal{T}})_{[\![r_x]\!]}$

of $M_{\mathcal{T}}$ has the tridiagonal form

$$(M_{\mathcal{T}})_{[\![r_x]\!]} = \sqrt{d}\begin{pmatrix} 0 & \sqrt{\alpha_x} & & & & \\ \sqrt{\alpha_x} & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & 1 & 0 & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & 0 \end{pmatrix} \tag{3.3}$$

(see Lemma B.1 below). For $\alpha_x > 1$, we define the vector $\mathbf{u} = (u_k)_{k=0}^{N-1}$ with components

$$u_1 := \left(\frac{\alpha_x}{\alpha_x - 1}\right)^{1/2} u_0, \qquad u_i := \left(\frac{1}{\alpha_x - 1}\right)^{(i-1)/2} u_1, \qquad u_j = 0$$

for $i = 2, 3, \ldots, r_x$ and $j = r_x + 1, \ldots, N-1$. If $\alpha_x > 2$ then u_i decays exponentially with i. Therefore, using the tridiagonal structure of $(M_{\mathcal{T}})_{[\![r_x]\!]}$ from (3.3) and that r_x is large, we see that $\mathbf{u}$ is an approximate eigenvector of $M_{\mathcal{T}}$ corresponding to the approximate eigenvalue $\sqrt{d}\Lambda(\alpha_x)$, where $\Lambda(t)$ is defined as in (2.2) (see Lemma C.1 below). Therefore, owing to (3.2), the vector

$$\sum_{i=0}^{r_x} u_i \mathbf{s}_i \tag{3.4}$$

is an approximate eigenvector of $A_{\mathcal{T}}$ with approximate eigenvalue $\sqrt{d}\Lambda(\alpha_x)$.

For $i = 0, \ldots, r_x$, we have $\mathbf{s}_i = |S_i^{\mathcal{T}}(x)|^{-1/2}\mathbf{1}_{S_i^{\mathcal{T}}(x)}$ (see Lemma B.1 below). Hence, the construction in (3.4) naturally suggests to consider

$$\mathbf{v} = \sum_{i=0}^{r_x} u_i |S_i(x)|^{-1/2}\mathbf{1}_{S_i(x)} \tag{3.5}$$

as approximate eigenvector of A, i.e., to replace $\mathbf{s}_i$ in (3.4) by $|S_i(x)|^{-1/2}\mathbf{1}_{S_i(x)}$. In Proposition 4.1 below, we show that $\mathbf{v}$ is an approximate eigenvector of A with approximate eigenvalue $\sqrt{d}\Lambda(\alpha_x)$. The proof heavily relies on the properties (a), (b) and (c) listed above and justified in Section 4.

The proof of Theorem 2.1 requires two additional key steps. Namely,

(i) two different vertices of large degree induce two different eigenvalues,

(ii) all eigenvalues of modulus larger than $2\sqrt{d}$ arise from vertices of large degree.

We remark that (i) is equivalent to a lower bound on the lth largest eigenvalue in terms of the lth largest degree of G while (ii) is equivalent to a corresponding upper bound.

For (i), we construct a subgraph G_2 of G such that A is well approximated by the adjacency matrix A_2 of G_2 and $B_{r_x}^{G_2}(x)$ and $B_{r_y}^{G_2}(y)$ are disjoint if $x, y \in [N]$, $x \neq y$ and $\alpha_x, \alpha_y \geqslant 2$ (see Lemma 6.2 below). Hence, the construction in (3.5) yields two orthogonal approximate eigenvectors of A which thus induce two different eigenvalues (or the same eigenvalue with multiplicity at least two). This completes (i) (cf. Proposition 6.1).

Thanks to (i), we now know that $\lambda_1(\underline{A}) \geqslant \ldots \geqslant \lambda_L(\underline{A}) \geqslant (2 + o(1))\sqrt{d}$ if $L := N - |V|$ and $V := \{x \in [N] \colon \alpha_x \leqslant 2\}$. Hence, (ii) will follow if we can show that $\lambda_{L+1}(\underline{A}) \leqslant (2 + o(1))\sqrt{d}$. By the min-max principle, we have

$$\max_{\mathbf{w} \in \mathbb{S}(U)} \langle \mathbf{w}, \underline{A}\mathbf{w} \rangle \geqslant \lambda_{L+1}(\underline{A}),$$

where $\mathbb{S}(U)$ is the unit sphere in the linear subspace $U := \text{span}\{\mathbf{1}_x : x \in V\} \subset \mathbb{R}^N$. Thus, it suffices to establish an upper bound on the largest eigenvalue μ of $\underline{A}_V$. This will be deduced from the matrix inequality

$$I_N + D + o(1) \geqslant d^{-1/2}\underline{A} \tag{3.6}$$

which holds with very high probability. Here, $D = (\alpha_x \delta_{xy})_{x,y\in[N]}$ is the diagonal matrix of normalized degrees. The inequality (3.6) is a consequence of an estimate on the nonbacktracking matrix associated with $\underline{A}$ and an Ihara-Bass type formula from [3].

We now explain how to prove that μ is at most $(2+o(1))\sqrt{d}$. Let $\tilde{\mathbf{w}} = (\tilde{w}_x)_{x\in V}$ be a normalized eigenvector of $\underline{A}_V$ corresponding to μ. We define a normalized vector $\mathbf{w} = (w_x)_{x\in[N]} \in \mathbb{R}^N$ through $w_x = \tilde{w}_x$ for $x \in V$ and $w_x = 0$ for $x \in [N] \setminus V$. Since $\langle \tilde{\mathbf{w}}, \underline{A}_V \tilde{\mathbf{w}}\rangle = \langle \mathbf{w}, \underline{A}\mathbf{w}\rangle$ we can evaluate the inequality in (3.6) at $\mathbf{w}$. This yields

$$\frac{\mu}{\sqrt{d}} - o(1) \leqslant \langle \mathbf{w}, (I_N + D)\mathbf{w}\rangle = 1 + \sum_{x\colon \alpha_x > 2} \alpha_x |w_x|^2 + \sum_{x\colon 2\geqslant \alpha_x > \tau} \alpha_x |w_x|^2 + \sum_{x\colon \alpha_x \leqslant \tau} \alpha_x |w_x|^2 \tag{3.7}$$

for any $\tau \in (1,2)$, where we used that $\mathbf{w}$ is normalized. The contribution for $\alpha_x > 2$ vanishes as $w_x = 0$ for such x. Since $\mathbf{w}$ is normalized the contribution for $\alpha_x \leqslant \tau$ is at most τ. We choose $\tau = 1 + o(1)$.

What remains is estimating the sum in the regime $2 \geqslant \alpha_x > \tau$. In the following paragraph, we shall sketch the proof of the bound

$$|w_x|^2 \leqslant \varepsilon \|\mathbf{w}|_{B_{r_x}^{G_\tau}(x)}\|^2 \tag{3.8}$$

which holds for some $\varepsilon = o(1)$ uniformly for all $x \in [N]$ satisfying $\tau < \alpha_x \leqslant 2$. Here, G_τ is a subgraph of G such that $A_\tau = \text{Adj}(G_\tau)$, the adjacency matrix of G_τ, and A are close and $B_{r_x}^{G_\tau}(x)$ and $B_{r_y}^{G_\tau}(y)$ are disjoint for all vertices $x, y \in [N]$ satisfying $x \neq y$ and $\alpha_x, \alpha_y > \tau$ (compare Lemma 6.2 below). Given (3.8), we conclude

$$\sum_{x\colon 2\geqslant\alpha_x>\tau} \alpha_x |w_x|^2 \leqslant 2 \sum_{x\colon 2\geqslant\alpha_x>\tau} |w_x|^2 \leqslant 2\varepsilon \|\mathbf{w}\|^2,$$

where we employed in the last step that $(\mathbf{1}_{B_{r_x}^{G_\tau}(x)})_{x\colon \alpha_x>\tau}$ is a family of orthogonal vectors. Since $\|\mathbf{w}\| = 1$, $\varepsilon = o(1)$ and $\tau = 1 + o(1)$, we obtain from (3.7) that $\mu \leqslant (1 + o(1) + \tau + 2\varepsilon)\sqrt{d} = (2 + o(1))\sqrt{d}$. Therefore, $\lambda_{L+1}(\underline{A}) \leqslant \mu \leqslant (2 + o(1))\sqrt{d}$.

We now sketch the proof of (3.8). For the graph $\mathcal{T}$ described above, the delocalization estimate in (3.8) can be obtained by analysing the tridiagonal matrix $M_{\mathcal{T}}$ introduced in (3.2) via a transfer matrix argument. As G_τ is close to $\mathcal{T}$ locally around a vertex x satisfying $\alpha_x > \tau$ the tridiagonal matrix $\widehat{M}$ constructed from A_τ around x is well approximated by $M_{\mathcal{T}}$. Hence, the transfer matrices associated with $\widehat{M}$ and $M_{\mathcal{T}}$ are also close and a version of the argument for $M_{\mathcal{T}}$ can be used to deduce (3.8). This completes the sketch of the proof of (ii) and thus the sketch of the proof of Theorem 2.1.

4. Large eigenvalues induced by vertices of large degree

Let G be an Erdős-Rényi graph on the vertex set $[N]$ with edge probability d/N. Let $A = \text{Adj}(G)$ be the adjacency matrix of G and $\underline{A} := A - \mathbb{E}A$. Proposition 4.1 below, the main result of this section, shows that each vertex of sufficiently large degree induces two approximate eigenvectors of $\underline{A}$. As explained after the statement of Proposition 4.1, this locates a positive and a negative eigenvalue of $\underline{A}$ of large modulus.

We now introduce the notation necessary to define the approximate eigenvectors. To lighten notation, we fix the vertex x throughout and omit all arguments (x) from our notation. In particular, we just write S_i and B_i instead of $S_i(x)$ and $B_i(x)$. Define

$$r_x := \left\lfloor \frac{\log N}{3 \log D_x} \right\rfloor, \tag{4.1}$$

and let $r \leqslant r_x$. Let $u_0 > 0$ and define the coefficients

$$u_1 := \frac{\sqrt{D_x}}{\sqrt{D_x - d}} u_0, \qquad u_i := \frac{d^{(i-1)/2}}{(D_x - d)^{(i-1)/2}} u_1 \quad (i = 2, 3, \dots, r+1). \tag{4.2}$$

Here and in the following, we exclusively consider the event $\{D_x > d\}$ such that $u_1, \dots, u_{r+1}$ are always well-defined. Then, on the event $S_i \neq \emptyset$ for $i = 1, \dots, r$, we define the approximate eigenvectors $\mathbf{v} \equiv \mathbf{v}(x, r)$ and $\mathbf{v}_- \equiv \mathbf{v}_-(x, r)$ through

$$\mathbf{v} := \sum_{i=0}^{r} u_i \mathbf{s}_i, \qquad \mathbf{v}_- := \sum_{i=0}^{r} (-1)^i u_i \mathbf{s}_i, \qquad \mathbf{s}_i := |S_i|^{-1/2} \mathbf{1}_{S_i}. \tag{4.3}$$

Finally, we choose u_0 so that the normalization $\|\mathbf{v}\|^2 = \|\mathbf{v}_-\|^2 = \sum_{i=0}^{r} u_i^2 = 1$ holds.

For the following proposition, we recall the definition $\alpha_x = D_x/d$.

Proposition 4.1 (Eigenvectors induced by vertex of large degree)**.** *Let $x \in [N]$ be a fixed vertex. Suppose that $\log d \leqslant r \leqslant r_x$. Then*

$$\|(\underline{A} - \sqrt{d}\Lambda(\alpha_x))\mathbf{v}\| + \|(\underline{A} + \sqrt{d}\Lambda(\alpha_x))\mathbf{v}_-\| \leqslant \mathcal{C}\bigg(\log d + \frac{\log N}{d}\bigg)^{1/2}\bigg(1 + \frac{\log N}{D_x}\bigg)^{1/2}$$

with very high probability on

$$\bigg\{\bigg(2 + 2\frac{\log d}{r}\bigg)d \leqslant D_x \leqslant \sqrt{N}(2d)^{-r}\bigg\} \cap \bigg\{D_x \geqslant \mathcal{K}\frac{\log N}{d}\bigg\} \tag{4.4}$$

conditioned on D_x, where $\mathcal{K} \geqslant 1$ is a constant that is chosen large enough depending on ν in the definition of very high probability.

We remark that if M is a Hermitian matrix and $\mathbf{v}$ a normalized vector such that $\|M\mathbf{v}\| \leqslant \varepsilon$ then M has an eigenvalue in $[-\varepsilon, \varepsilon]$. Therefore, Proposition 4.1 implies that $\underline{A}$ possesses with very high probability two eigenvalues $\lambda_\pm$ in the vicinity of $\pm\sqrt{d}\Lambda(\alpha_x)$ if α_x is sufficiently large.

We shall show in Lemma 4.4 below that $S_i \neq \emptyset$ for $i = 1, \dots, r$ with very high probability on the event

$$\bigg\{\mathcal{K}\frac{\log N}{d} \leqslant D_x \leqslant \sqrt{N}(2d)^{-r}\bigg\}. \tag{4.5}$$

Here and in the following, $\mathcal{K}$ denotes a constant depending only on ν in the definition of very high probability.

To prove Proposition 4.1, we shall decompose $(\underline{A} - \sqrt{d}\Lambda(\alpha_x))\mathbf{v}$ into a sum $\mathbf{w}_0 + \cdots + \mathbf{w}_4$ of vectors, which are all proved to have a small norm. (See Lemma 4.2 below and the estimates in Lemma 4.3 below.) Each of the vectors $\mathbf{w}_i$ will turn out to be small for a different reason, which is why we treat them individually.

In order to define the vectors $\mathbf{w}_i$, we introduce the notations

$$\mathbf{e} := N^{-1/2}\mathbf{1}_{[N]}, \qquad N_i(y) := \langle \mathbf{1}_y , A\mathbf{1}_{S_i} \rangle = |S_i \cap S_1(y)| \tag{4.6}$$

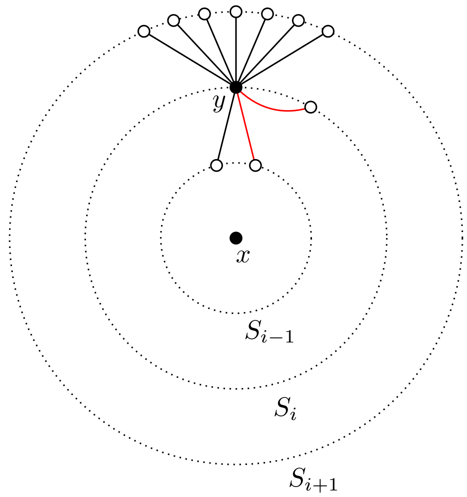

Figure 4.1. An illustration of the definition of $N_i(y)$ from (4.6), where $y \in S_i(x)$. The red edges are forbidden in a tree. For a tree, $N_{i+1}(y) = D_y - 1$, $N_i(y) = 0$, and $N_{i-1}(y) = 1$.

for all $i = 0, \dots, r$ and $y \in [N]$. Thus, $N_i(y)$ is the number of edges starting in S_i and ending in y. Note that if the graph $G|_{B_{i+1}}$ is a tree then it is easy to see that

$$N_i(y) = \mathbb{1}_{y \in S_{i-1}}(D_y - \mathbb{1}_{i \geqslant 2}) + \mathbb{1}_{y \in S_{i+1}} \tag{4.7}$$

with the convention that $S_{-1} := \emptyset$. See Figure 4.1 for an illustration of $N_i(y)$.

Define

$$\begin{aligned}
\mathbf{w}_0 &:= \frac{d}{N}\mathbf{v} - d\langle \mathbf{e}, \mathbf{v}\rangle \mathbf{e}, \\
\mathbf{w}_1 &:= \sum_{i=0}^{r} \frac{u_i}{\sqrt{|S_i|}} \left(\sum_{y \in S_{i+1}} (N_i(y) - 1)\mathbf{1}_y + \sum_{y \in S_i} N_i(y)\mathbf{1}_y \right), \\
\mathbf{w}_2 &:= \sum_{i=1}^{r} \frac{u_i}{\sqrt{|S_i|}} \sum_{y \in S_{i-1}} \left(N_i(y) - \frac{|S_i|}{|S_{i-1}|} \right) \mathbf{1}_y, \\
\mathbf{w}_3 &:= u_2 \left(\frac{\sqrt{|S_2|}}{\sqrt{|S_1|}} - \sqrt{d} \right) \mathbf{s}_1 + \sum_{i=2}^{r-1} \left[u_{i+1} \left(\frac{\sqrt{|S_{i+1}|}}{\sqrt{|S_i|}} - \sqrt{d} \right) + u_{i-1} \left(\frac{\sqrt{|S_i|}}{\sqrt{|S_{i-1}|}} - \sqrt{d} \right) \right] \mathbf{s}_i, \\
\mathbf{w}_4 &:= \left(u_{r-1} \frac{\sqrt{|S_r|}}{\sqrt{|S_{r-1}|}} - u_{r-1}\sqrt{d} - u_{r+1}\sqrt{d} \right) \mathbf{s}_r + u_r \frac{\sqrt{|S_{r+1}|}}{\sqrt{|S_r|}} \mathbf{s}_{r+1}\,.
\end{aligned} \tag{4.8}$$

Lemma 4.2 (Decomposition of $(\underline{A} - \sqrt{d}\Lambda(\alpha_x))\mathbf{v}$)**.** *We have the decomposition*

$$(\underline{A} - \sqrt{d}\Lambda(\alpha_x))\mathbf{v} = \mathbf{w}_0 + \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3 + \mathbf{w}_4. \tag{4.9}$$

Lemma 4.2 will be shown in Subsection 4.1 below. We now explain the origin and interpretation of the different errors $\mathbf{w}_0, \dots, \mathbf{w}_4$.

- The vector $\mathbf{w}_0$ is equal to $-(\mathbb{E}A)\mathbf{v}$, and hence takes care of the expectation $\mathbb{E}A$ in the definition of $\underline{A} = A - \mathbb{E}A$. It will turn out to be small because the vector $\mathbf{v}$ is localized near the vertex x, and hence has a small overlap with $\mathbf{e}$, which is completely delocalized.

- The vector $\mathbf{w}_1$ quantifies the extent to which $G|_{B_{r+1}}$ deviates from a tree. Indeed, by (4.7) it vanishes if $G|_{B_{r+1}}$ is a tree. It will turn out to be small because the number of cycles in $G|_{B_{r+1}}$ is not too large.
- The vector $\mathbf{w}_2$ quantifies the extent to which $G|_{B_r}$ deviates from a tree with the property that, for each $i \geqslant 2$, all vertices in S_i have the same degree. Indeed, it is immediate that the term $i = 1$ is always zero, and the other terms vanish under the above condition, by (4.7). It will turn out to be small because the number of cycles in $G|_{B_{r+1}}$ is not too large and because the $N_i(y)$ will concentrate around $\frac{|S_i|}{|S_{i-1}|}$ for most vertices $y \in S_{i-1}$, for any $i \geqslant 2$.
- The vector $\mathbf{w}_3$ quantifies the extent to which $G|_{B_r}$ deviates from a tree that is d-regular at all vertices in $B_r \setminus \{x\}$. It will turn out to be small because, with very high probability, most vertices in $B_r \setminus \{x\}$ will have degree close to d.
- Finally, the vector $\mathbf{w}_4$ quantifies the error arising from edges connecting the ball B_r, where the tree approximation is valid, to the rest of the graph $[N] \setminus B_r$, where it is not. It will be small by the exponential decay of the coefficients u_i.

Lemma 4.3 (Estimates on $\mathbf{w}_0, \dots, \mathbf{w}_4$)**.** *For any $r \leqslant r_x$, the estimates*

$$\|\mathbf{w}_0\| = \mathcal{O}(dN^{-1/4}) \tag{4.10a}$$

$$\|\mathbf{w}_1\| = \mathcal{O}(d^{-1/2}), \tag{4.10b}$$

$$\|\mathbf{w}_2\| = \mathcal{O}\bigg(\bigg(\log d + \frac{\log N}{d}\bigg)^{1/2}\bigg(1 + \frac{\log N}{D_x}\bigg)^{1/2}\bigg), \tag{4.10c}$$

$$\|\mathbf{w}_3\| = \mathcal{O}\bigg(\bigg(\frac{\log N}{D_x}\bigg)^{1/2}\bigg), \tag{4.10d}$$

$$\|\mathbf{w}_4\| = \mathcal{O}\bigg(\bigg(\frac{d}{D_x - d}\bigg)^{(r-2)/2}\bigg(\frac{d}{\sqrt{D_x - d}} + \bigg(\frac{\log N}{D_x}\bigg)^{1/2}\bigg)\bigg) \tag{4.10e}$$

hold with very high probability on $\{D_x > d\} \cap \{\mathcal{K} \log N/d \leqslant D_x \leqslant \sqrt{N}(2d)^{-r}\}$ conditioned on $A_{(x)}$.

Lemma 4.3 is proved in Subsection 4.2 below.

Proof of Proposition 4.1. From Lemmas 4.2 and 4.3 we get

$$\|(\underline{A} - \sqrt{d}\Lambda(\alpha_x))\mathbf{v}\| \leqslant \mathcal{C}\bigg(\log d + \frac{\log N}{d}\bigg)^{1/2}\bigg(1 + \frac{\log N}{D_x}\bigg)^{1/2} + \mathcal{C}\sqrt{d}\bigg(\frac{d}{D_x - d}\bigg)^{(r-1)/2}$$

with very high probability on $\{D_x > 2d\} \cap \{\mathcal{K} \log N/d \leqslant D_x \leqslant \sqrt{N}(2d)^{-r}\}$. Write $D_x = (2+t)d$ for $t > 0$. To conclude the proof, it suffices to show that

$$\sqrt{d}\bigg(\frac{d}{D_x - d}\bigg)^{(r-1)/2} \leqslant 1 \qquad \text{i.e.} \qquad \log(1+t) \geqslant \frac{\log d}{r-1}\,. \tag{4.11}$$

Since by assumption $\log d \leqslant r$, this condition is satisfied provided that

$$2\frac{\log d}{r} \leqslant t = \frac{D_x}{d} - 2\,.$$

This concludes the proof for $\mathbf{v}$. For $\mathbf{v}_-$ the bound follows in the same way from trivial modifications of Lemma 4.2 and Lemma 4.3 obtained by replacing u_i by $(-1)^i u_i$. We leave the details of these modifications to the reader. □

4.1. Decomposition of $(\underline{A}-\sqrt{d}\Lambda(\alpha_x))\mathbf{v}$ – Proof of Lemma 4.2. In this subsection, we prove Lemma 4.2. We recall that $\mathbf{w}_0, \ldots, \mathbf{w}_4$ were defined in (4.8) and $N_i(y)$ in (4.6).

Proof of Lemma 4.2. Recalling the definition $\underline{A} = A - \mathbb{E}A$, we have

$$\underline{A}\mathbf{v} = A\mathbf{v} + \frac{d}{N}\mathbf{v} - d\langle \mathbf{e}, \mathbf{v}\rangle \mathbf{e} = \mathbf{w}_0 + \sum_{i=0}^{r} \frac{u_i}{\sqrt{|S_i|}} A\mathbf{1}_{S_i}.$$

By the definition (4.6) of $N_i(y)$ and the triangle inequality for the graph distance, we have

$$A\mathbf{1}_{S_i} = \mathbb{1}_{i\geqslant 1} \sum_{y\in S_{i-1}} N_i(y)\mathbf{1}_y + \sum_{y\in S_i} N_i(y)\mathbf{1}_y + \sum_{y\in S_{i+1}} N_i(y)\mathbf{1}_y,$$

so that

$$\begin{aligned}
\underline{A}\mathbf{v} &= \mathbf{w}_0 + \sum_{i=1}^{r} \frac{u_i}{\sqrt{|S_i|}} \sum_{y\in S_{i-1}} N_i(y)\mathbf{1}_y + \sum_{i=0}^{r} \frac{u_i}{\sqrt{|S_i|}} \Bigg[\sum_{y\in S_i} N_i(y)\mathbf{1}_y + \sum_{y\in S_{i+1}} N_i(y)\mathbf{1}_y\Bigg] \\
&= \mathbf{w}_0 + \mathbf{w}_1 + \sum_{i=1}^{r} \frac{u_i}{\sqrt{|S_i|}} \Bigg[\sum_{y\in S_{i-1}} N_i(y)\mathbf{1}_y + \mathbf{1}_{S_{i+1}}\Bigg] + u_0\mathbf{1}_{S_1} \\
&= \mathbf{w}_0 + \mathbf{w}_1 + \mathbf{w}_2 + \sum_{i=1}^{r} \frac{u_i}{\sqrt{|S_i|}} \Bigg[\sum_{y\in S_{i-1}} \frac{|S_i|}{|S_{i-1}|}\mathbf{1}_y + \mathbf{1}_{S_{i+1}}\Bigg] + u_0\mathbf{1}_{S_1}.
\end{aligned}$$

Thus, we conclude

$$\begin{aligned}
\underline{A}\mathbf{v} - \sum_{k=0}^{2}\mathbf{w}_k &= u_0\mathbf{1}_{S_1} + \sum_{i=1}^{r} \frac{u_i}{\sqrt{|S_i|}} \Bigg[\frac{|S_i|}{|S_{i-1}|}\mathbf{1}_{S_{i-1}} + \mathbf{1}_{S_{i+1}}\Bigg]. \\
&= u_0\sqrt{|S_1|}\,\mathbf{s}_1 + u_1\sqrt{|S_1|}\,\mathbf{s}_0 + u_2\frac{\sqrt{|S_2|}}{\sqrt{|S_1|}}\,\mathbf{s}_1 + \sum_{i=2}^{r-1}\Bigg(u_{i+1}\frac{\sqrt{|S_{i+1}|}}{\sqrt{|S_i|}} + u_{i-1}\frac{\sqrt{|S_i|}}{\sqrt{|S_{i-1}|}}\Bigg)\mathbf{s}_i \\
&\quad + u_r\frac{\sqrt{|S_{r+1}|}}{\sqrt{|S_r|}}\,\mathbf{s}_{r+1} + u_{r-1}\frac{\sqrt{|S_r|}}{\sqrt{|S_{r-1}|}}\,\mathbf{s}_r.
\end{aligned} \tag{4.12}$$

Since $\sqrt{d}\Lambda(\alpha_x) = \frac{D_x}{\sqrt{D_x - d}}$, from the definition of u_i in (4.2) we get

$$\sqrt{d}\Lambda(\alpha_x)u_0 = \sqrt{D_x}u_1, \qquad \sqrt{d}\Lambda(\alpha_x)u_1 = \sqrt{D_x}u_0 + \sqrt{d}u_2, \qquad \sqrt{d}\Lambda(\alpha_x)u_i = \sqrt{d}u_{i-1} + \sqrt{d}u_{i+1}$$

for all $i = 2, 3, \ldots, r$. This implies

$$\sqrt{d}\Lambda(\alpha_x)\mathbf{v} = u_1\sqrt{|S_1|}\,\mathbf{s}_0 + \Big(u_0\sqrt{|S_1|} + u_2\sqrt{d}\Big)\mathbf{s}_1 + \sum_{i=2}^{r}\Big(u_{i-1}\sqrt{d} + u_{i+1}\sqrt{d}\Big)\mathbf{s}_i.$$

Together with (4.12), this yields

$$\begin{aligned}
&(\underline{A} - \sqrt{d}\Lambda(\alpha_x))\mathbf{v} - \sum_{k=0}^{2}\mathbf{w}_k \\
&= u_2\Bigg(\frac{\sqrt{|S_2|}}{\sqrt{|S_1|}} - \sqrt{d}\Bigg)\mathbf{s}_1 + \sum_{i=2}^{r-1}\Bigg[u_{i+1}\Bigg(\frac{\sqrt{|S_{i+1}|}}{\sqrt{|S_i|}} - \sqrt{d}\Bigg) + u_{i-1}\Bigg(\frac{\sqrt{|S_i|}}{\sqrt{|S_{i-1}|}} - \sqrt{d}\Bigg)\Bigg]\mathbf{s}_i \\
&\quad + \Bigg(u_{r-1}\frac{\sqrt{|S_r|}}{\sqrt{|S_{r-1}|}} - u_{r-1}\sqrt{d} - u_{r+1}\sqrt{d}\Bigg)\mathbf{s}_r + u_r\frac{\sqrt{|S_{r+1}|}}{\sqrt{|S_r|}}\,\mathbf{s}_{r+1} \\
&= \mathbf{w}_3 + \mathbf{w}_4,
\end{aligned}$$

which concludes the proof. ∎

4.2. Smallness of $\mathbf{w}_0$, ..., $\mathbf{w}_4$ – Proof of Lemma 4.3. This subsection is devoted to the proof of Lemma 4.3.

In order to estimate $\mathbf{w}_0, \dots, \mathbf{w}_4$, we will make frequent use of the following lemma.

Lemma 4.4 (Concentration of $|S_i|$)**.** *(i) For $r, i \in \mathbb{N}$ satisfying $1 \leqslant i \leqslant r$ we have*

$$\left| \frac{|S_{i+1}|}{d|S_i|} - 1 \right| = \mathcal{O}\bigg(\Big(\frac{\log N}{d|S_i|} \Big)^{1/2} \bigg), \tag{4.13a}$$

and

$$|S_i| = D_x d^{i-1} \bigg(1 + \mathcal{O}\bigg(\Big(\frac{\log N}{dD_x} \Big)^{1/2} \bigg) \bigg) \tag{4.13b}$$

with very high probability on $\{\mathcal{K} \log N/d \leqslant D_x \leqslant \sqrt{N}(2d)^{-r}\}$ conditioned on $A_{(x)}$.

(ii) Moreover, for all $r, i \in \mathbb{N}$ satisfying $1 \leqslant i \leqslant r$, the bound

$$|S_{i+1}| \leqslant d|S_i| + \mathcal{C}(d|S_i| \log N)^{1/2} \tag{4.14}$$

holds with very high probability on $\{D_x \leqslant \sqrt{N}(2d)^{-r}\}$ conditioned on $A_{(x)}$.

Before proving Lemma 4.4, we first conclude (4.10a), (4.10d) and (4.10e) from it.

Proof of (4.10a)*,* (4.10d) *and* (4.10e). In the whole proof, we exclusively work on the event $\{D_x > d\} \cap \{\mathcal{K} \log N/d \leqslant D_x \leqslant \sqrt{N}(2d)^{-r}\}$. For the proof of (4.10a), we start by using the Cauchy-Schwarz inequality to obtain

$$\|\mathbf{w}_0\| \leqslant \frac{d}{N} + \frac{d}{\sqrt{N}} \sum_{y \in B_r} |\mathbf{v}(y)| \leqslant \frac{d}{N} + \frac{d}{\sqrt{N}} \sqrt{|B_r|}. \tag{4.15}$$

From Lemma 4.4, we conclude with very high probability

$$|B_r| \leqslant 1 + \sum_{i=0}^{r-1} D_x (2d)^i \leqslant 2D_x(2d)^{r-1} \leqslant (2D_x)^r \leqslant (2D_x)^{r_x} \leqslant \sqrt{N}\,,$$

by definition of r_x. Hence (4.10a) follows.

We now turn to the proof of (4.10d). Estimating the definition of $\mathbf{w}_3$ yields

$$\begin{aligned}
\|\mathbf{w}_3\|^2 &\leqslant d \left[\left(\frac{\sqrt{|S_2|}}{\sqrt{d|S_1|}} - 1 \right)^2 u_2^2 + 2 \sum_{i=2}^{r-1} \left(\left(\frac{\sqrt{|S_{i+1}|}}{\sqrt{d|S_i|}} - 1 \right)^2 u_{i+1}^2 + \left(\frac{\sqrt{|S_i|}}{\sqrt{d|S_{i-1}|}} - 1 \right)^2 u_{i-1}^2 \right) \right] \\
&\leqslant \mathcal{C} \frac{\log N}{D_x} \left[u_2^2 + 2 \sum_{i=2}^{r-1} (u_{i+1}^2 + u_{i-1}^2) \right]
\end{aligned}$$

with very high probability conditioned on $A_{(x)}$, where we used (4.13a) and (4.13b) in the last step. This completes the proof of (4.10d).

Next, we apply the triangle inequality and Lemma 4.4 to $\mathbf{w}_4$ to obtain

$$\begin{aligned}
\|\mathbf{w}_4\| &\leqslant |u_{r-1}| \left| \frac{\sqrt{|S_r|}}{\sqrt{|S_{r-1}|}} - \sqrt{d} \right| + |u_{r+1}|\sqrt{d} + |u_r| \frac{\sqrt{|S_{r+1}|}}{\sqrt{|S_r|}} \\
&\leqslant |u_{r-1}| \mathcal{C} \Big(\frac{\log N}{D_x} \Big)^{1/2} + |u_r|\sqrt{d} \\
&\leqslant \frac{d^{(r-2)/2}}{(D_x - d)^{(r-2)/2}} \bigg(\frac{d}{\sqrt{D_x - d}} + \mathcal{C} \Big(\frac{\log N}{D_x} \Big)^{1/2} \bigg),
\end{aligned}$$

where we used the definition of u_{r-1} and u_r as well as $|u_1| \leqslant 1$ in the last step. This verifies (4.10e). □

Proof of Lemma 4.4. Define

$$\mathcal{E}_i := \frac{d|S_i|}{N} + \frac{1}{\sqrt{N}}\,.$$

We shall prove below that there are constants $C, c > 0$ such that if $|B_i| \leqslant \sqrt{N}$ for some $i \geqslant 1$ and $\varepsilon \in [0,1]$, then

$$\mathbb{P}\Big((1-\varepsilon - C\mathcal{E}_i)d|S_i| \leqslant |S_{i+1}| \leqslant (1+\varepsilon + C\mathcal{E}_i)d|S_i| \,\Big|\, A_{(B_{i-1})}\Big) \geqslant 1 - 2\exp\Big(-cd|S_i|\varepsilon^2\Big). \tag{4.16}$$

From (4.16), we now conclude (4.13a) and

$$D_x\Big(\frac{d}{2}\Big)^{i-1} \leqslant |S_i| \leqslant D_x(2d)^{i-1}\,, \tag{4.17}$$

for $1 \leqslant i \leqslant r$ simultaneously by induction. For $i = 1$ we choose $\varepsilon^2 = \mathcal{C}\log N/(dD_x)$, which is bounded by 1 for $\mathcal{K}$ in (4.5) large enough. Using that $\mathcal{E}_1 \leqslant \varepsilon$, we obtain (4.13a) directly from (4.16). The estimate (4.17) is trivial for $i = 1$.

For the induction step, we assume that $\mathcal{K}$ in (4.5) is large enough that the right-hand side of (4.13a) for $i = 1$ is less than $1/2$. Suppose first that with very high probability (4.13a) and (4.17) hold up to i. Since $|S_i| \geqslant D_x$ by $(4.17)_i$, we conclude from $(4.13\text{a})_i$ that $(4.17)_{i+1}$ holds. Next, suppose that with very high probability (4.13a) holds up to i and (4.17) up to $i+1$. By $(4.17)_{i+1}$, we deduce that for $i+1 \leqslant r$ we have $|B_{i+1}| \leqslant D_x(2d)^r \leqslant \sqrt{N}$, where the last inequality follows by assumption on D_x. Hence, we may apply $(4.16)_{i+1}$ to estimate $|S_{i+2}|$, with the choice $\varepsilon^2 = \mathcal{C}\log N/(d|S_{i+1}|)$ with the same $\mathcal{C}$ as in the first induction step. It is easy to check that $\mathcal{E}_{i+1} \leqslant \varepsilon$, and hence we conclude $(4.13\text{a})_{i+1}$ after taking the conditional expectation with respect to $A_{(x)}$.

The expansion in (4.13b) is a direct consequence of

$$D_x d^{i-1}(1-\varepsilon_i) \leqslant |S_i| \leqslant D_x d^{i-1}(1+\varepsilon_i), \qquad \varepsilon_i := 2\mathcal{C}\Big(\frac{\log N}{dD_x}\Big)^{1/2}\sum_{j=0}^{i-2} d^{-j/2}, \tag{4.18}$$

for $1 \leqslant i \leqslant r$ with very high probability on $\{\mathcal{K}\log N/d \leqslant D_x \leqslant \sqrt{N}(2d)^{-r}\}$ conditioned on $A_{(x)}$ as well as the fact that the geometric sum in the definition of ε_i is bounded by 2 uniformly in i.

The estimates in (4.13a) and (4.17) imply (4.18) by induction as follows. The case $i = 1$ is trivial. For the induction step, we conclude from (4.13a) that

$$|S_{i+1}| \geqslant d|S_i|\bigg(1 - \mathcal{C}\Big(\frac{\log N}{d|S_i|}\Big)^{1/2}\bigg) \geqslant D_x d^i\bigg(1 - \varepsilon_i - \mathcal{C}\Big(\frac{\log N}{d^i D_x(1-\varepsilon_i)}\Big)^{1/2}\bigg).$$

Here, we used $|S_i| \neq 0$ by (4.17) in the first step and the induction hypothesis in the second step. As $\varepsilon_i \leqslant 3/4$ for sufficiently large $\mathcal{K}$ due to $D_x \geqslant \mathcal{K}\log N/d$ we obtain the lower bound in (4.18). The upper bound is proved completely analogously. This completes the proof of (4.13b).

Next, we also conclude (4.14) from (4.16) by an induction argument. For $i = 1$, we can assume that $D_x = |S_1| \neq 0$ as there is nothing to show otherwise. If $|S_1| \neq 0$ then we choose $\varepsilon^2 = \mathcal{C}\log N/(d|S_1|)$ in (4.16). As $\mathcal{E}_1 \leqslant \varepsilon$ this directly implies (4.14) for $i = 1$. In the induction step we assume $|S_{i+1}| \neq 0$ as $(4.14)_{i+1}$ is trivial for $|S_{i+1}| = 0$. The induction hypothesis and $D_x \leqslant \sqrt{N}(2d)^{-r}$ imply $|B_{i+1}| \leqslant \sqrt{N}$. Hence, $|S_{i+1}| \neq 0$ allows the choice $\varepsilon^2 = \mathcal{C}\log N/(d|S_{i+1}|)$ in (4.16) in order to bound $|S_{i+2}|$. As $\mathcal{E}_{i+1} \leqslant \varepsilon$ we obtain $(4.14)_{i+1}$ by taking the conditional expectation with respect to $A_{(x)}$.

What remains therefore is the proof of (4.16). We condition on $A_{(B_{i-1})}$ and suppose that $|B_i| \leqslant \sqrt{N}$. (Note that B_i is a measurable function of $A_{(B_{i-1})}$.) Let us compute the law of

$|S_{i+1}|$ conditioned on $A_{(B_{i-1})}$. For $y \in B_i^c$ denote by $F_y = \mathbb{1}_{y \text{ adjacent to } S_i}$. Then, conditioned on $A_{(B_{i-1})}$, $(F_y)_{y\in B_i^c}$ are i.i.d. Bernoulli random variables with expectation $1-(1-p)^{|S_i|}$, where $p = d/N$. Thus,

$$|S_{i+1}| = \sum_{y\in B_i^c} F_y \overset{\mathrm{d}}{=} \operatorname{Binom}\Big(1-(1-p)^{|S_i|}, N - |B_i|\Big)$$

conditioned on $A_{(B_{i-1})}$. Thus,

$$E_{i+1} := \mathbb{E}[|S_{i+1}||A_{(B_{i-1})}] = (1-(1-p)^{|S_i|})(N-|B_i|) = d|S_i|(1+O(\mathcal{E}_i)) \tag{4.19}$$

where we used that $d|S_i| \leqslant d\sqrt{N} \leqslant N$ by the assumption $|B_i| \leqslant \sqrt{N}$. Applying Bennett's inequality yields

$$\begin{aligned}\mathbb{P}\Big(\big||S_{i+1}| - E_{i+1}\big| > \varepsilon\, E_{i+1} \,\Big|\, A_{(B_{i-1})}\Big) &\leqslant 2\exp\Big(-E_{i+1}\min\{h(1+\varepsilon), h(1-\varepsilon)\}\Big)\\ &\leqslant 2\exp\Big(-\frac{1}{2}d|S_i|\min\{h(1+\varepsilon), h(1-\varepsilon)\}\Big)\,.\end{aligned}$$

where we used (4.19) and that $\mathcal{E}_i = O(N^{-1/4})$ as follows from the assumptions $|B_i| \leqslant \sqrt{N}$ and $(2d)^{r+1} \leqslant 2\sqrt{N}$. Now the claim (4.16) follows from (4.19) and the observation that there exists a $c > 0$ such that $\min\{h(1+\varepsilon), h(1-\varepsilon)\} \geqslant c\varepsilon^2$ for all $\varepsilon \in [0,1]$. □

Lemma 4.5 (Few cycles in small balls)**.** *For $k, r \in \mathbb{N}$ we have*

$$\mathbb{P}\Big(|E(G|_{B_r})| - |B_r| + 1 \geqslant k \,\Big|\, S_1\Big) \leqslant \frac{1}{N^k}\big(C(d+|S_1|)\big)^{2kr+k}(2kr)^{2k}\,. \tag{4.20}$$

Remark 4.6. Using the definition (4.1), it is easy to check that for any $\nu > 0$ there exists a $k \in \mathbb{N}$ such that the right-hand side of (4.20) is bounded by $N^{-\nu}$ provided that $r \leqslant r_x$ and $D_x > d$. Hence, Lemma 4.5 says that the number of cycles k in $G|_{B_r}$ is with very high probability bounded on $\{D_x > d\} \cap \{r \leqslant r_x\}$ conditioned on S_1.

Corollary 4.7. *With very high probability on $\{D_x > d\}$ conditioned on S_1, we have for all $i \leqslant r_x + 1$*

$$|S_i| = \sum_{y\in S_i} N_{i-1}(y) + \mathcal{O}(1), \tag{4.21a}$$

$$|S_i| = \sum_{y\in S_{i-1}} N_i(y) + \mathcal{O}(1), \tag{4.21b}$$

as well as

$$\sum_{y\in S_i} N_i(y) = \mathcal{O}(1) \tag{4.22}$$

Proof. By choosing a spanning tree of $G|_{B_{r_x}}$, we conclude that, with very high probability on $\{D_x > d\}$ conditioned on S_1, we can find $\mathcal{O}(1)$ edges of $G|_{B_{r_x}}$ such that removing them yields a tree on the vertex set B_{r_x}. Now (4.21a) and (4.21b) follow easily by noting that

$$\langle \mathbf{1}_{S_{i-1}}, A\mathbf{1}_{S_i}\rangle = \sum_{y\in S_{i-1}} N_i(y) = \sum_{y\in S_i} N_{i-1}(y)\,,$$

and that the left-hand side equals $|S_i|$ if A is the adjacency matrix of a tree. Finally, (4.22) follows by noting that its left-hand side vanishes if A is the adjacency matrix of a tree. □

Before the proof of Lemma 4.5, we show how Lemma 4.5 and (4.21a) are used to bound $\|\mathbf{w}_1\|$ and establish (4.10b).

Proof of (4.10b)*.* Since $S_0 = \{x\}$ we have that $N_0(y) = 1$ for all $y \in S_1$. Moreover, $N_0(x) = 0$ since G has no loops. Hence,

$$\mathbf{w}_1 = \sum_{i=1}^{r} \frac{u_i}{\sqrt{|S_i|}} \biggl(\sum_{y \in S_{i+1}} (N_i(y) - 1)\mathbf{1}_y + \sum_{y \in S_i} N_i(y)\mathbf{1}_y \biggr). \tag{4.23}$$

By the triangle inequality, $|u_i| \leqslant 1$, and the fact that $N_i(y) \geqslant 1$ for all $y \in S_{i+1}$, we find

$$\|\mathbf{w}_1\| \leqslant \sum_{i=1}^{r} \frac{1}{\sqrt{|S_i|}} \biggl(\sum_{y \in S_{i+1}} (N_i(y) - 1) + \sum_{y \in S_i} N_i(y) \biggr) = \sum_{i=1}^{r} \frac{1}{\sqrt{|S_i|}} \mathcal{O}(1)\,,$$

where in the last step we used (4.21a) and (4.22). The claim follows using (4.13b). □

Proof of Lemma 4.5. Throughout the proof we condition on S_1. Let $r \leqslant N$ and $k \in \mathbb{N}$, and without loss of generality $r, k \geqslant 1$. Define the set $\mathcal{H}_k$ as the set of connected graphs H satisfying $x \in V(H) \subset [N]$, $S_1^H \subset S_1$, $|E(H)| = |V(H)| - 1 + k$, and $|V(H)| \leqslant 2kr + 1$. Here S_1^H denotes the unit sphere around x of the graph H. Let $H \in \mathcal{H}_k$. Then

$$\mathbb{P}(E(H) \subset E(G) \,|\, S_1) = \biggl(\frac{d}{N}\biggr)^{|E(H)| - |S_1^H|} = \biggl(\frac{d}{N}\biggr)^{|V(H)| - 1 + k - |S_1^H|}.$$

Hence, by a union bound,

$$\mathbb{P}(\exists H \in \mathcal{H}_k,\, E(H) \subset E(G) \,|\, S_1) \leqslant \sum_{H \in \mathcal{H}_k} \mathbb{P}(E(H) \subset E(G) \,|\, S_1)\,.$$

In the sum over $H \in \mathcal{H}_k$, we shall sum first over the set of vertices S_1^H, the set of vertices $V(H) \setminus (S_1^H \cup \{x\})$, and then over all graphs H on the vertex set $V(H)$. Writing $u_1 := |S_1^H|$ and $u_2 := |V(H) \setminus (S_1^H \cup \{x\})|$, so that $|V(H)| = u_1 + u_2 + 1$, we find

$$\mathbb{P}(\exists H \in \mathcal{H}_k,\, E(H) \subset E(G) \,|\, S_1) \leqslant \sum_{0 \leqslant u_1 + u_2 \leqslant 2kr} \binom{|S_1|}{u_1} \binom{N - |S_1| - 1}{u_2} C_{u_1 + u_2 + 1, k} \biggl(\frac{d}{N}\biggr)^{u_2 + k},$$

where $C_{u,k}$ is the number of connected graphs on u vertices with $u - 1 + k$ edges. To estimate $C_{u,k}$, we note that each such graph can be written as a union of a tree on u vertices and k additional edges. By Cayley's theorem on the number of trees, we therefore conclude that

$$C_{u,k} \leqslant u^{u-2} u^{2k} = u^{u + 2k - 2}\,.$$

Putting everything together, we conclude that

$$\begin{aligned} \mathbb{P}(\exists H \in \mathcal{H}_k,\, E(H) \subset E(G) \,|\, S_1) &\leqslant \sum_{0 \leqslant u_1 + u_2 \leqslant 2kr} \frac{|S_1|^{u_1}}{u_1!} \frac{N^{u_2}}{u_2!} (u_1 + u_2 + 1)^{u_1 + u_2 + 2k - 1} \biggl(\frac{d}{N}\biggr)^{u_2 + k} \\ &\leqslant \frac{1}{N^k} \bigl(C(d + |S_1|)\bigr)^{2kr + k} (2kr)^{2k}\,. \end{aligned}$$

In order to conclude the proof, it suffices to show that

$$\bigl\{|E(G|_{B_r})| - |B_r| + 1 \geqslant k\bigr\} \subset \bigl\{\exists H \in \mathcal{H}_k,\, E(H) \subset E(G)\bigr\}\,. \tag{4.24}$$

To show (4.24), we suppose that G is in the event on the left-hand side of (4.24). Let T be a spanning tree of B_r such that $d_T(x,y) = d(x,y)$ for all $y \in B_r$, where d_T is the graph distance on T. Since $|E(G|_{B_r})| - |B_r| + 1 \geqslant k$, we can find k edges of $G|_{B_r}$ that are not edges of T; denote these edges by E_1. Let U_1 denote the vertices incident to the edges of E_1. Let U_2 denote the vertices in all the (unique) paths of T connecting the vertices of U_1 to x, and E_2 the edges of these paths. Consider now the graph H with vertex set $U_1 \cup U_2 \cup \{x\}$ and edge set $E_1 \cup E_2$. See Figure 4.2 for an illustration.

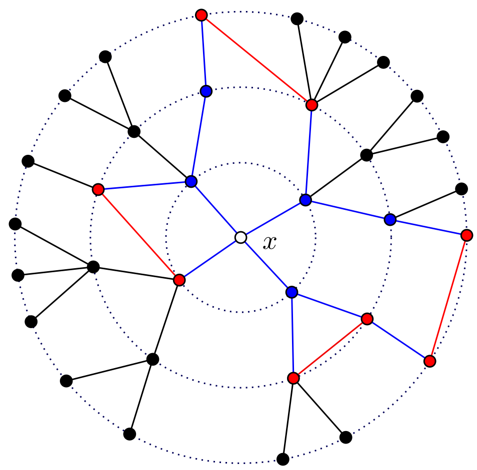

Figure 4.2. Graphical representation of the proof of (4.24). We draw the ball B_r for $r = 3$. The spanning tree T is drawn using black and blue edges, and the spheres of radii $1, 2, 3$ are drawn using dots. The red edges are E_1, the blue edges E_2, the red vertices U_1, and the blue vertices $U_2 \setminus U_1$.

We claim that $H \in \mathcal{H}_k$, which will conclude the proof of (4.24). The only non-obvious property to verify is that $|V(H)| = |U_1 \cup U_2 \cup \{x\}| \leqslant 2kr + 1$. This follows easily from the observation that $|U_1| \leqslant 2k$ and that each of the above paths has at most $r - 1$ vertices in $[N] \setminus (U_1 \cup \{x\})$, and there are at most $2k$ paths. This concludes the proof. $\square$

Proof of (4.10c). We define $\Xi_i \coloneqq \{D_x > d\} \cap \{\mathcal{K} \log N/d \leqslant D_x \leqslant \sqrt{N}(2d)^{-i}\}$ for $i \in \mathbb{N}$. We start by noting that

$$\mathbf{w}_2 = \sum_{i=2}^{r} \frac{u_i}{\sqrt{|S_i|}} \sum_{y \in S_{i-1}} \left(N_i(y) - \frac{|S_i|}{|S_{i-1}|} \right) \mathbf{1}_y,$$

since $S_0 = \{x\}$ and $N_1(x) = |S_1| = |S_1|/|S_0|$.

We apply the Pythagorean theorem, use (4.21b), $|S_i| \geqslant D_x d/2$ uniformly for $i \in \{2, \ldots, r\}$ with very high probability on Ξ_r conditioned on $A_{(x)}$ by (4.13b) in Lemma 4.4 as well as

$\sum_{i=1}^{r-1} u_{i+1}^2 \leqslant 1$ and obtain

$$\begin{aligned}
\|\mathbf{w}_2\|^2 &= \sum_{i=1}^{r-1} \frac{u_{i+1}^2}{|S_{i+1}|} \sum_{y \in S_i} \biggl(N_{i+1}(y) - \frac{|S_{i+1}|}{|S_i|}\biggr)^2 \\
&\leqslant 2 \sum_{i=1}^{r-1} \frac{u_{i+1}^2}{|S_{i+1}|} \sum_{y \in S_i} \biggl(N_{i+1}(y) - \frac{1}{|S_i|} \sum_{y \in S_i} N_{i+1}(y)\biggr)^2 + \frac{\mathcal{C}}{D_x d} \\
&\leqslant \frac{4}{d} \max_{1 \leqslant i \leqslant r-1} (Z_i + Y_i^2) + \frac{\mathcal{C}}{D_x d}\,,
\end{aligned}$$

where we defined

$$Z_i := \frac{1}{|S_i|} \sum_{y \in S_i} \Bigl(N_{i+1}(y) - \mathbb{E}[N_{i+1}(y) \,|\, A_{(B_{i-1})}]\Bigr)^2, \qquad Y_i := \frac{1}{|S_i|} \sum_{y \in S_i} \Bigl(N_{i+1}(y) - \mathbb{E}[N_{i+1}(y) \,|\, A_{(B_{i-1})}]\Bigr).$$

Here we used Young's inequality and the fact that $\mathbb{E}[N_{i+1}(y) \,|\, A_{(B_{i-1})}]$ does not depend on $y \in S_i$. In fact, conditioned on $A_{(B_{i-1})}$, the random variables $(N_{i+1}(y))_{y \in S_i}$ are i.i.d. with law $\mathrm{Binom}(N - |B_i|, d/N)$.

The term Y_i can be easily estimated by Bennett's inequality for $\mathrm{Binom}(|S_i|(N - |B_i|), d/N)$, which yields

$$|Y_i| \leqslant \mathcal{C} \frac{\log N}{D_x}$$

with very high probability on Ξ_r conditioned on $A_{(x)}$. Here we used that, by Lemma 4.4, $N - |B_i| \geqslant N/2$ and $|S_i| \geqslant D_x$, and, by definition of Ξ_r, $\frac{\log N}{d D_x} \leqslant 1$.

What remains, therefore, is the estimate of Z_i. We shall prove that for all $1 \leqslant i \leqslant r-1$

$$|Z_i| \leqslant \mathcal{C} d \biggl(1 + \frac{\log N}{|S_i|}\biggr) \biggl(\log d + \frac{\log N}{d}\biggr) \tag{4.25}$$

with very high probability on Ξ_i conditioned on $A_{(x)}$, which will conclude the proof of the lemma.

The estimate (4.25) can be regarded as a concentration result for the degrees of the vertices in S_i; indeed, by Lemma 4.5 for any $y \in S_i$ we have $D_y = N_{i+1}(y) + \mathcal{O}(1)$ with very high probability. For any vertex $y \in S_i$ we have the variance estimate $\sqrt{\mathbb{E}(D_y - d)^2} \asymp C\sqrt{d}$. On the other hand, in the relevant regime $d \leqslant C \log N$, the estimate with very high probability (following from Bennett's inequality) $|D_y - d| \leqslant \mathcal{C} \log N$ is much worse. Essentially, we need an estimate with very high probability of the average $\frac{1}{|S_i|} \sum_{y \in S_i} (D_y - d)^2$, and the trivial bound $\mathcal{C}(\log N)^2$ obtained by applying the above estimate is much too large. Instead, we need to use that the typical term of Z_i is much smaller than $(\log N)^2$. We do this using a dyadic classification of the degrees of the vertices in S_i.

For the proof of (4.25), we always condition on $A_{(x)}$ and work on the event Ξ_i. We abbreviate

$$E_y := N_{i+1}(y) - \mathbb{E}[N_{i+1}(y) \,|\, A_{(B_{i-1})}]$$

for $y \in S_i$, and introduce the level set sizes

$$L_s^i := |\{y \in S_i \colon E_y^2 > s^2 d^2\}| \tag{4.26}$$

for any $s > 0$. We have the probabilistic tail bound on L_s^i

$$\mathbb{P}(L_s^i \geqslant \ell \,|\, A_{(B_{i-1})}) \leqslant \binom{|S_i|}{\ell} \exp(-c d \ell (s \wedge s^2)) \tag{4.27}$$

for all $s > 0$ and $\ell \in \mathbb{N}$, with very high probability. Here $c > 0$ is a universal constant. To prove (4.27), we use a union bound to get

$$
\begin{aligned}
\mathbb{P}(L_s^i \geqslant \ell \mid A_{(B_{i-1})}) &\leqslant \sum_{T \subset S_i,\ |T|=\ell} \mathbb{P}\Big(E_y^2 > (sd)^2 \text{ for all } y \in T \ \Big|\ A_{(B_{i-1})}\Big) \\
&\leqslant \binom{|S_i|}{\ell} \max_{T \subset S_i,\ |T|=\ell} \mathbb{P}\Big(E_y^2 > (sd)^2 \text{ for all } y \in T \ \Big|\ A_{(B_{i-1})}\Big) \qquad (4.28) \\
&= \binom{|S_i|}{\ell} \mathbb{P}\Big(E_y^2 > (sd)^2 \ \Big|\ A_{(B_{i-1})}\Big)^\ell
\end{aligned}
$$

for some $y \in S_i$, since $(N_{i+1}(y))_{y \in S_i}$ are i.i.d. conditioned on $A_{(B_{i-1})}$. By Bennett's inequality, we obtain

$$\mathbb{P}\Big(E_y^2 > (sd)^2 \ \Big|\ A_{(B_{i-1})}\Big) \leqslant \exp(-cd(s \wedge s^2))$$

since $N - |B_i| \geqslant N/2$ (by Lemma 4.4), and hence (4.27) follows.

Next we conclude the argument by establishing (4.25). We decompose

$$|S_i| Z_i = \sum_{y \in S_i} E_y^2 \leqslant d|S_i| + \sum_{y \in S_i} E_y^2 \mathbb{1}_{E_y^2 \geqslant d}. \qquad (4.29)$$

In order to estimate the second summand in (4.29), we now establish the dyadic decomposition

$$\sum_{y \in S_i} E_y^2 \mathbb{1}_{E_y^2 \geqslant d} \leqslant \sum_{k=k_{\min}}^{k_{\max}} \sum_{y \in \mathcal{N}_k^i} E_y^2 \leqslant d^2 \sum_{k=k_{\min}}^{k_{\max}} \mathrm{e}^{k+1} |\mathcal{N}_k^i| \qquad (4.30)$$

with very high probability conditioned on $A_{(B_{i-1})}$, where we introduced

$$
\begin{aligned}
\mathcal{N}_k^i &:= \Big\{ y \in S_i \colon d^2 \mathrm{e}^k < E_y^2 \leqslant \mathrm{e}^{k+1} d^2 \Big\}, \qquad k \in \mathbb{Z}, \\
k_{\min} &:= -\lceil \log(d) \rceil, \\
k_{\max} &:= \left\lceil \mathcal{C} + \log\left(\frac{\log N}{d}\right) \vee \left(2 \log\left(\frac{\log N}{d}\right)\right) \right\rceil
\end{aligned}
$$

with some possibly ν-dependent constant $\mathcal{C} > 0$.

We now prove (4.30) by showing that, for all $y \in S_i$, $E_y^2 \leqslant d^2 \mathrm{e}^{k_{\max}+1}$ with very high probability conditioned on $A_{(B_{i-1})}$. To that end, we note that

$$\mathbb{P}\Big(\exists y \in S_i \,, E_y^2 > d^2 \mathrm{e}^{k_{\max}+1} \ \Big|\ A_{(B_{i-1})}\Big) = \mathbb{P}(L_s^i \geqslant 1 \,|\, A_{(B_{i-1})}) \,, \qquad s = \mathrm{e}^{(k_{\max}+1)/2} \,.$$

Moreover, from (4.27) with $\ell = 1$ and $|S_i| \leqslant N$ we obtain

$$\mathbb{P}(L_s^i \geqslant 1 \,|\, A_{(B_{i-1})}) \leqslant N \exp(-cd(s \wedge s^2)) \leqslant N^{-\nu} \,,$$

where the last inequality follows for $s = \mathrm{e}^{(k_{\max}+1)/2}$ and $k_{\max}$ defined above. This establishes (4.30).

Next, we estimate $|\mathcal{N}_k^i|$. This will allow us to conclude the statement of the lemma from (4.29) and (4.30). In fact, we have $\mathbb{P}(|\mathcal{N}_k^i| \geqslant \ell \mid A_{(B_{i-1})}) \leqslant \mathbb{P}(L^i_{\mathrm{e}^{k/2}} \geqslant \ell \mid A_{(B_{i-1})})$. We choose $\ell = \ell_k$, where

$$\ell_k := \frac{\mathcal{C}}{d}(|S_i| + \log N) \begin{cases} \mathrm{e}^{-k/2}, & \text{if } k \geqslant 0, \\ \mathrm{e}^{-k}, & \text{if } k < 0. \end{cases}$$

Using (4.27) and estimating $\binom{|S_i|}{\ell} \leqslant \mathrm{e}^{|S_i|}$, we deduce that $|\mathcal{N}_k^i| \leqslant \ell_k$ with very high probability on Ξ_i conditioned on $A_{(x)}$.

With this information, we now estimate the right-hand side of (4.30). We conclude from (4.29) and (4.30) that

$$\begin{aligned}
|S_i|Z_i &\leqslant d|S_i| + d^2 \sum_{k=k_{\min}}^{k_{\max}} \mathrm{e}^{k+1}\ell_k \\
&\leqslant d|S_i| + \mathcal{C}d(|S_i| + \log N)\Bigg(\sum_{k=k_{\min}}^{0} \mathrm{e}^{k+1}\mathrm{e}^{-k} + \mathbb{1}_{k_{\max}\geqslant 0}\sum_{k=0}^{k_{\max}} \mathrm{e}^{k+1}\mathrm{e}^{-k/2}\Bigg) \\
&\leqslant d|S_i| + \mathcal{C}d(|S_i| + \log N)\Big(|k_{\min}| + \mathrm{e}^{k_{\max}/2}\Big) \\
&\leqslant \mathcal{C}d(|S_i| + \log N)\Bigg(\log(d) + \bigg(\frac{\log N}{d}\bigg)^{1/2} \vee \frac{\log N}{d}\Bigg),
\end{aligned}$$

which concludes the proof of (4.25) and hence also of (4.10c). □

5. Quadratic form estimates on centred adjacency matrix

The main result of the present section is a bound on $\underline{A} = A - \mathbb{E}A$ in Proposition 5.1 below. In the following, we write $S \geqslant T$ for two Hermitian $N \times N$ matrices S, T if $S - T$ is positive semidefinite, i.e., $\langle \mathbf{w}, S\mathbf{w}\rangle \geqslant \langle \mathbf{w}, T\mathbf{w}\rangle$ for all $\mathbf{w} \in \mathbb{R}^N$. We remark that Δ was defined in (3.1).

Proposition 5.1 (Upper bound on $d^{-1/2}\underline{A}$)**.** *If $4 \leqslant d \leqslant N^{2/13}$ then, with very high probability, we have*

$$I_N + D + E \geqslant d^{-1/2}|\underline{A}|,$$

where $|\underline{A}| = \sqrt{\underline{A}^*\underline{A}}$, D is the diagonal matrix defined through $D := \operatorname{diag}(D_x/d)_{x\in[N]}$ and the error matrix E satisfies

$$\|E\| \leqslant \frac{\mathcal{C}(d+\Delta)}{d^{3/2}} = \mathcal{C}d^{-1/2}\begin{cases} 1 + d^{-1/2}\sqrt{\log N}, & \text{if } d \geqslant \frac{1}{2}\log N, \\ 1 + \frac{\log N}{d(\log\log N - d)}, & \text{if } d \leqslant \frac{1}{2}\log N \end{cases}$$

with very high probability.

We postpone the proof of Proposition 5.1 to Section 5.1 below. First we state and prove the following corollary of Proposition 5.1.

Corollary 5.2 (Norm bound on $\underline{A}$)**.** *If $4 \leqslant d \leqslant N^{2/13}$ then we have*

$$\|\underline{A}\| \leqslant \sqrt{d} + \frac{\Delta}{\sqrt{d}} = \begin{cases} 2\sqrt{d} + \mathcal{C}\sqrt{\log N} & \text{if } d \geqslant \frac{1}{2}\log N, \\ \sqrt{d} + \mathcal{C} + \mathcal{C}\frac{\log N}{d^{1/2}(\log\log N - \log d)} & \text{if } d \leqslant \frac{1}{2}\log N \end{cases}$$

with very high probability.

Corollary 1.3 in [2] and Corollary 3.3 in [3] provide similar statements to Corollary 5.2.

Proof of Corollary 5.2. Owing to Lemma 3.3, we have

$$\|D\| \leqslant \frac{\Delta}{d} = \begin{cases} 1 + \mathcal{C}d^{-1/2}\sqrt{\log N} & \text{if } d \geqslant \frac{1}{2}\log N, \\ \mathcal{C}\frac{\log N}{d(\log\log N - \log d)} & \text{if } d \leqslant \frac{1}{2}\log N \end{cases}$$

with very high probability. Therefore, Corollary 5.2 follows immediately from Proposition 5.1. □

5.1. Proof of Proposition 5.1. Let B be the nonbacktracking matrix associated to $d^{-1/2}\underline{A} = d^{-1/2}(A - \mathbb{E}A)$, i.e., the $N^2 \times N^2$ matrix with entries

$$B_{ef} := d^{-1/2}\underline{A}_{kl}\mathbb{1}_{j=k}\mathbb{1}_{i\neq l} \tag{5.1}$$

for $e = (i, j) \in [N]^2$ and $f = (k, l) \in [N]^2$. The next proposition provides a high probability bound on the spectral radius of the nonbacktracking matrix. It is proved as Theorem 2.5 in [3].

Proposition 5.3 (Bound on the nonbacktracking matrix of $d^{-1/2}\underline{A}$)**.** *There are universal constants $C > 0$ and $c > 0$ such that, for all $4 \leqslant d \leqslant N^{2/13}$ and $\varepsilon \geqslant 0$, we have*

$$\mathbb{P}\big(\rho(B) \geqslant 1 + \varepsilon\big) \leqslant CN^{3-c\sqrt{d}\log(1+\varepsilon)}.$$

The Ihara-Bass-type formula in the following lemma relates the spectrum of B and the spectrum of $\underline{A}$. Its formulation is identical to the one of Lemma 4.1 in [3]. Therefore, we shall not present its proof in this paper.

Lemma 5.4 (Ihara-Bass-type formula)**.** *Let B be the nonbacktracking matrix associated to $d^{-1/2}\underline{A}$ as in (5.1) and let $t \in \mathbb{C}$ satisfy $t^2 \neq d^{-1}\underline{A}_{ij}\underline{A}_{ji}$ for all $i, j \in [N]$. We define the matrices $\underline{A}(t) = (\underline{A}_{ij}(t))_{i,j=1}^N$ and $M(t) = (m_i(t)\delta_{ij})_{i,j=1}^N$ through*

$$\underline{A}_{ij}(t) := \frac{t\sqrt{d}\underline{A}_{ij}}{t^2 d - \underline{A}_{ij}\underline{A}_{ji}}, \qquad m_i(t) := 1 + \sum_k \frac{\underline{A}_{ik}\underline{A}_{ki}}{t^2 d - \underline{A}_{ik}\underline{A}_{ki}}.$$

Then $t \in \operatorname{spec}(B)$ if and only if $\det(M(t) - \underline{A}(t)) = 0$.

An argument similar to the following proof of Proposition 5.1 has been used to show Proposition 4.2 in [3].

Proof of Proposition 5.1. We only show that $d^{-1/2}\underline{A} \leqslant I_N + D + E$. The same proof implies that $-\underline{A}$ satisfies the same bound. In this proof, we use the matrices $\underline{A}(t)$ and $M(t)$ defined in Lemma 5.4 exclusively for $t \in \mathbb{R}$. Note that $\underline{A}(t)$ and $M(t)$ are Hermitian for all $t \in \mathbb{R}$. If $t \in \mathbb{R}$ converges to $+\infty$ then we have that $M(t) - \underline{A}(t) \to I_N$. Therefore, $M(t) - \underline{A}(t)$ is strictly positive definite for all sufficiently large $t > 0$. Let t_* be the infimum of all $t > 0$ such that $M(t) - \underline{A}(t)$ is strictly positive definite. Hence, the smallest eigenvalue of $M(t_*) - \underline{A}(t_*)$ is zero while all eigenvalues of $M(t) - \underline{A}(t)$ are strictly positive for $t > t_*$. Therefore, Lemma 5.4 implies that $t_* \in \operatorname{spec}(B)$ and $M(t) - \underline{A}(t)$ is strictly positive definite for all $t \in (\rho(B), \infty)$. Hence, Proposition 5.3 yields that

$$\mathbb{P}\big(M(1+\varepsilon) - \underline{A}(1+\varepsilon) \geqslant 0\big) \geqslant 1 - CN^{3-c\sqrt{d}\log(1+\varepsilon)}. \tag{5.2}$$

We shall show below that, for each $t \in [1, 2]$, we have

$$\|\underline{A}(t) - t^{-1}d^{-1/2}\underline{A}\| \leqslant \frac{\mathcal{C}\Delta}{d^{3/2}} = \frac{\mathcal{C}}{d^{3/2}}\begin{cases} d + \sqrt{d\log N}, & \text{if } d \geqslant \frac{1}{2}\log N, \\ \frac{\log N}{\log\log N - d}, & \text{if } d \leqslant \frac{1}{2}\log N, \end{cases} \tag{5.3a}$$

$$\|M(t) - I_N - t^{-2}D\| \leqslant \frac{\mathcal{C}\Delta}{d^2} = \frac{\mathcal{C}}{d^2}\begin{cases} d + \sqrt{d\log N}, & \text{if } d \geqslant \frac{1}{2}\log N, \\ \frac{\log N}{\log\log N - d}, & \text{if } d \leqslant \frac{1}{2}\log N, \end{cases} \tag{5.3b}$$

with very high probability. Since $M(1+\varepsilon) - \underline{A}(1+\varepsilon) \geqslant 0$ and $D \geqslant 0$ imply

$$d^{-1/2}\underline{A} \leqslant I_N + D + \varepsilon + (1+\varepsilon)\big(\|\underline{A}(t) - t^{-1}d^{-1/2}\underline{A}\| + \|M(t) - I_N - t^{-2}D\|\big),$$

choosing $\varepsilon = \mathcal{C} d^{-1/2}$ and using (5.2), (5.3) as well as $\log(1+\varepsilon) \geqslant c\varepsilon$ for some $c > 0$ establish Proposition 5.1 up to showing (5.3).

We now prove (5.3). In order to estimate $\underline{A}(t) - t^{-1} d^{-1/2} \underline{A}$, we use its Hermiticity to conclude

$$\|\underline{A}(t) - t^{-1} d^{-1/2} \underline{A}\| \leqslant \max_i \sum_j |\underline{A}_{ij}(t) - t^{-1} d^{-1/2} \underline{A}_{ij}|.$$

A short computation shows that

$$\max_i \sum_j |\underline{A}_{ij}(t) - t^{-1} d^{-1/2} \underline{A}_{ij}| \leqslant \max_i \sum_j \frac{|\underline{A}_{ij}|^3}{t\sqrt{d}(t^2 d - \underline{A}_{ij}^2)} \leqslant \frac{2}{t^3 d^{3/2}} \big(\max_i D_i + 1 \big),$$

where we used $|A_{ij}| \leqslant 1$, $t^2 d/2 \geqslant \underline{A}_{ij}^2$ and $\sum_j \underline{A}_{ij}^2 \leqslant D_i + 1$ in the last step. Thus, $t \geqslant 1$ and Lemma 3.3 yield (5.3a).

As $M(t)$ and D are diagonal matrices by definition, we obtain

$$\|M(t) - I_N - t^{-2} D\| = \max_i \bigg| \sum_j \bigg(\frac{\underline{A}_{ij}^2}{t^2 d - \underline{A}_{ij}^2} - \frac{1}{t^2 d} A_{ij}^2 \bigg) \bigg| \leqslant \max_i \sum_j \bigg(\frac{\underline{A}_{ij}^4}{t^2 d (t^2 d - \underline{A}_{ij}^2)} + \frac{2 \underline{A}_{ij}^2}{t^2 N} + \frac{d}{t^2 N^2} \bigg).$$

Arguing similarly as in the proof of (5.3a) implies (5.3b). This completes the proof of (5.3) and, thus, the one of Proposition 5.1. □

6. Lower bounds on large eigenvalues

The main result of this section is the following proposition. It states that the lth largest eigenvalue of $\underline{A}$, $\lambda_l(\underline{A})$, is bounded from below by $\sqrt{d}\Lambda(\alpha_{\sigma(l)})$, up to a small error term, as long as $\alpha_{\sigma(l)}$ is sufficiently large. Here, the permutation σ of $[N]$ is chosen such that $(\alpha_{\sigma(l)})_{l=1}^N$ is nonincreasing. We recall that $\alpha_x := D_x/d$ for any $x \in [N]$ and the permutation σ is chosen such that $\alpha_{\sigma(1)} \geqslant \alpha_{\sigma(2)} \geqslant \ldots \geqslant \alpha_{\sigma(N)}$ (cf. (2.3)). Similarly, up to a small error term, $-\sqrt{d}\Lambda(\alpha_{\sigma(l)})$ bounds the lth smallest eigenvalue, $\lambda_{N+1-l}(\underline{A})$, of $\underline{A}$ from above.

Proposition 6.1. *Let $d \leqslant N^{2/13}$. There is a universal constant $C > 0$ such that if the random index $L_\geqslant$ is defined through*

$$L_\geqslant := \max \Big\{ l \in [N] \colon \alpha_{\sigma(l)} \geqslant \tau_* \Big\}, \qquad \tau_* := \bigg(2 + C \frac{(\log d)^2}{d \wedge \log N} \bigg) \vee \bigg(\mathcal{K} \frac{\log N}{d^2} \bigg) \tag{6.1}$$

then, for any $l \in [L_\geqslant]$, the bound

$$\min\{\lambda_l(\underline{A}), -\lambda_{N+1-l}(\underline{A})\} \geqslant \sqrt{d}\Lambda(\alpha_{\sigma(l)}) + \mathcal{O}\Bigg(\bigg(\log d + \frac{\log N}{d} \bigg)^{1/2} \bigg(1 + \frac{\log N}{D_{\sigma(l)}} \bigg)^{1/2} + \bigg(\frac{ld}{N \log d} \bigg)^{1/2} + \bigg(1 + \frac{\Delta}{d} \bigg) \bigg(\frac{\Delta}{D_{\sigma(l)}} \frac{d + \log N}{d} \bigg)^{1/2} \Bigg)$$

holds with very high probability. Here, Δ is defined as in (3.1). In the definition of $L_\geqslant$, we use the convention that $L_\geqslant = 0$ if $\alpha_x < \tau_*$ for all $x \in [N]$.

The following lemma will be a key ingredient in proof of the previous proposition. For its formulation, we introduce the set $\mathcal{V}_\tau$ of vertices of large degree given by

$$\mathcal{V}_\tau := \{ x \in [N] \colon D_x \geqslant \tau d \},$$

where $\tau > 1$. The following lemma provides a subgraph G_τ of G such that the shortest path in G_τ connecting two vertices in $\mathcal{V}_\tau$ has at least length $r+1$ for any $r \leqslant r_\tau$, where we defined

$$r_\tau := \frac{d}{2\log d} h\biggl(\frac{\tau - 1}{2}\biggr) - 2. \tag{6.2}$$

Lemma 6.2. *Let $\tau > 1$ and r_τ be defined as in (6.2). For all $x \in \mathcal{V}_\tau$, we set $r_{x,\tau} := (\frac{1}{4} r_x) \wedge (\frac{1}{2} r_\tau)$ with r_x from (4.1). Then there exists a subgraph G_τ of G with the following properties.*

(i) If a path p in G_τ connects two vertices $x, y \in \mathcal{V}_\tau$, $x \neq y$, then p has length at least $r_{x,\tau} + r_{y,\tau} + 1$. In particular, the balls $B^{G_\tau}_{r_{x,\tau}}(x)$ for $x \in \mathcal{V}_\tau$ are disjoint.

(ii) The induced subgraph $G_\tau|_{B^{G_\tau}_{r_{x,\tau}}(x)}$ is a tree for each $x \in \mathcal{V}_\tau$.

(iii) For each $x \in \mathcal{V}_\tau$ and each $i \in \mathbb{N}$ satisfying $1 \leqslant i \leqslant r_{x,\tau}$ we have $S^{G_\tau}_i(x) \subset S^G_i(x)$.

(iv) For each edge in $G \setminus G_\tau$, there is at least one vertex in $\mathcal{V}_\tau$ incident to it.

(v) The degrees induced on $[N]$ by $G \setminus G_\tau$ are bounded according to

$$\max_{x \in [N]} D^{G \setminus G_\tau}_x = \mathcal{O}\biggl(1 + \frac{\log N}{h((\tau - 1)/2)d}\biggr)$$

with very high probability.

(vi) Let $\mathcal{K} \log\log N \leqslant d \leqslant \mathcal{K}^{-1} N^{1/4}$. For each $x \in \mathcal{V}_\tau$ and all $2 \leqslant i \leqslant \log N/(4\log d)$, the bound

$$|S^G_i(x) \setminus S^{G_\tau}_i(x)| \leqslant D^{G \setminus G_\tau}_x d^{i-2} \Delta \biggl[1 + \mathcal{C}\sqrt{\frac{\log N}{d\Delta}}\biggr], \tag{6.3}$$

holds with very high probability. Here, Δ is defined as in (3.1).

We postpone the proof to the following subsection. First we now conclude Proposition 6.1 from Proposition 4.1, Corollary 5.2 and Lemma 6.2.

Proof of Proposition 6.1. We will only prove the statement about $\lambda_l(\underline{A})$ and leave the necessary modifications for the analogous statement about $\lambda_{N+1-l}(\underline{A})$ to the reader (see the proof of Proposition 4.1). For any $\tau > 1$, let G_τ be a subgraph of G possessing the properties described in Lemma 6.2.

We fix $l \in [L_\geqslant]$, set $x := \sigma(l)$ and $\tau = 2$. Let $\mathbf{v}^{(x)}$ be the associated approximate eigenvector of $\underline{A}$ around x constructed in (4.3) with $r = r_{x,2} := (\frac{1}{4} r_x) \wedge (\frac{1}{2} r_2)$ for $r_2 = r_{\tau=2}$ defined in (6.2). From Proposition 4.1, we conclude for $\lambda_x = \sqrt{d}\Lambda(\alpha_x)$ that

$$\|(\underline{A} - \lambda_x)\mathbf{v}^{(x)}\| = \mathcal{O}\biggl(\biggl(\log d + \frac{\log N}{d}\biggr)^{1/2}\biggl(1 + \frac{\log N}{D_x}\biggr)^{1/2}\biggr) \tag{6.4}$$

with very high probability. Here, we used $l \in [L_\geqslant]$ and (6.1) to conclude that $r_{x,\tau} \geqslant \log d$. Moreover, the lower bounds on D_x in (4.4) follow directly from the definition of $r = r_{x,2}$ and (6.1). The upper bound on D_x in (4.4) holds with very high probability due to Lemma 3.3. We define $\tilde{\mathbf{v}}^{(x)} := (\tilde{v}^{(x)}(y))_{y \in [N]}$ through

$$\tilde{v}^{(x)}(y) := v^{(x)}(y) \mathbb{1}_{y \in B^{G_\tau}_{r_{x,2}}(x)}.$$

We note that the vector $\tilde{\mathbf{v}}^{(x)}$ is not normalized. By the explicit definition of $\mathbf{v}^{(x)}$ in (4.3), we therefore conclude that

$$\|\mathbf{v}^{(x)} - \tilde{\mathbf{v}}^{(x)}\|^2 = \sum_{i=1}^{r_{x,2}} |u_i|^2 \frac{|S_i^G(x) \setminus S_i^{G_\tau}(x)|}{|S_i^G(x)|} = \mathcal{O}\bigg(\frac{\Delta}{D_x} \frac{D_x^{G \setminus G_\tau}}{d} \bigg)$$

with very high probability due to Lemma 6.2 (vi), (4.13b) combined with $\mathcal{K} \log N \leqslant D_x d$ by (6.1) and $\sum_{i=1}^{r_{x,2}} |u_i|^2 \leqslant 1$. Hence, we have that

$$\|\mathbf{v}^{(x)} - \tilde{\mathbf{v}}^{(x)}\| = \mathcal{O}\bigg(\bigg(\frac{\Delta}{D_x} \frac{D_x^{G \setminus G_\tau}}{d} \bigg)^{1/2} \bigg) \tag{6.5}$$

with very high probability. Therefore, from (6.5), Corollary 5.2, and (6.4), we deduce for $\lambda_x = \sqrt{d}\Lambda(\alpha_x)$ that

$$\begin{aligned} (\underline{A} - \lambda_x) \frac{\tilde{\mathbf{v}}^{(x)}}{\|\tilde{\mathbf{v}}^{(x)}\|} &= \mathcal{O}\bigg(\bigg(\log d + \frac{\log N}{d} \bigg)^{1/2} \bigg(1 + \frac{\log N}{D_x} \bigg)^{1/2} + \frac{d + \Delta}{\sqrt{d}} \bigg(\frac{\Delta}{D_x} \frac{D_x^{G \setminus G_\tau}}{d} \bigg)^{1/2} \bigg) \\ &= \mathcal{O}\bigg(\bigg(\log d + \frac{\log N}{d} \bigg)^{1/2} \bigg(1 + \frac{\log N}{D_x} \bigg)^{1/2} + \bigg(1 + \frac{\Delta}{d} \bigg) \bigg(\frac{\Delta}{D_x} \frac{d + \log N}{d} \bigg)^{1/2} \bigg) \end{aligned} \tag{6.6}$$

with very high probability. Here, we used Lemma 6.2 (v) in the last step. Hence, $(\tilde{\mathbf{v}}^{(\sigma(l))})_{l=1}^{L_\geqslant}$ defines a family of orthogonal approximate eigenvectors of $\underline{A}$ as their supports are disjoint by Lemma 6.2 (i).

We set $\widetilde{W}_l \coloneqq \operatorname{span}\{\tilde{\mathbf{v}}^{(\sigma(k))} \colon k \in [l]\}$. In the following, we write $\mathbb{S}(W)$ for the unit sphere with respect to the Euclidean norm in any linear subspace $W \subset \mathbb{R}^N$. The max-min principle for $\lambda_l(\underline{A})$ yields

$$\begin{aligned} \lambda_l(\underline{A}) &= \max_{\dim W = l} \min_{\mathbf{w} \in \mathbb{S}(W)} \langle \mathbf{w}, \underline{A}\mathbf{w} \rangle \\ &\geqslant \min_{\mathbf{w} \in \mathbb{S}(\widetilde{W}_l)} \langle \mathbf{w}, \underline{A}\mathbf{w} \rangle \\ &\geqslant \min_{\mathbf{w} \in \mathbb{S}(\widetilde{W}_l)} \langle \mathbf{w}, A_\tau \mathbf{w} \rangle - \|A - A_\tau\| - \|(\mathbb{E}A)|_{\widetilde{W}_l}\| \\ &\geqslant \min_{k \in [l]} \lambda_{\sigma(k)} - 2\|A - A_\tau\| - \|(\mathbb{E}A)|_{\widetilde{W}_l}\| - \max_{k \in [l]} \|(\underline{A} - \lambda_{\sigma(k)}) \tilde{\mathbf{v}}^{(x)}\| \|\tilde{\mathbf{v}}^{(x)}\|^{-1}. \end{aligned} \tag{6.7}$$

Here, we added and subtracted A_τ in the third step. The last step follows from the definition of $\widetilde{W}_l$ and the orthogonality of $(\tilde{\mathbf{v}}^{(\sigma(k))})_{k \in [l]}$.

Now, we estimate the terms on the right-hand side of (6.7) to obtain the lower bound on $\lambda_l(\underline{A})$ in the proposition. Since $t \mapsto \Lambda(t)$ is monotonically increasing for $t \geqslant 2$, the first term is bounded from below by $\sqrt{d}\Lambda(\alpha_{\sigma(l)})$. For the second term, we use $\|A - A_\tau\| \leqslant \max_{x \in [N]} D_x^{G \setminus G_\tau} \leqslant \mathcal{C}(1 + \log N / d)$ with very high probability by Lemma 6.2 (v). If $\mathbf{w} \in \mathbb{S}(\widetilde{W}_l)$ then $\mathbf{w}$ has at most $l \cdot r_\tau$ nonzero components. Hence,

$$\|(\mathbb{E}A)_{\widetilde{W}_l}\| \leqslant \frac{d}{N} + \sqrt{\frac{\mathcal{C}ld}{N \log d}}.$$

For the fourth term in (6.7), we use (6.6). This completes the proof of Proposition 6.1. □

6.1. Proof of Lemma 6.2. For the proof of Lemma 6.2 we need the next lemma. For any $x \in \mathcal{V}_\tau$, it provides a bound on the number of other vertices in $\mathcal{V}_\tau$ whose distance from x is sufficiently small.

Lemma 6.3. *Let $\tau > 1$ and let $r \in \mathbb{N}$ satisfy $r \leqslant r_\tau$ with r_τ from (6.2). Then, for any $x \in \mathcal{V}_\tau$, we have*

$$|\mathcal{V}_\tau \cap B_r(x)| = \mathcal{O}\biggl(\frac{\log N}{h((\tau-1)/2)d}\biggr) \tag{6.8}$$

with very high probability.

The following lemma controls the growth of $|S_i(z)|$ in terms of d and Δ. In contrast to (4.13b) in Lemma 4.4, no lower bound on D_z is required.

Lemma 6.4. *Let $\mathcal{K} \log\log N \leqslant d \leqslant \mathcal{K}^{-1} N^{1/4}$ and let $z \in [N]$. For any $i \leqslant \log N/(4\log d)$, the bound*

$$|S_i(z)| \leqslant \Delta d^{i-1}\biggl[1 + \mathcal{C}\biggl(\frac{\log N}{d\Delta}\biggr)^{1/2}\biggr]$$

holds with very high probability. Here, Δ is defined as in (3.1).

The proofs of the previous two lemmas are postponed until the end of this section.

Proof of Lemma 6.2. In the entire proof, we will write $\mathcal{V}$ instead of $\mathcal{V}_\tau$. We will construct a subgraph H_τ of G in two steps such that $G_\tau = G \setminus H_\tau$ satisfies the properties stated in the lemma. For a graphical depiction of the following argument, we refer to Figure 6.1.

First, we construct a subgraph $H^{(1)} \subset G$ such that $B_{r_x}^{G\setminus H^{(1)}}(x)$ is a tree for each $x \in \mathcal{V}$. Indeed, for any $x \in \mathcal{V}$ we apply the following algorithm. For each $y \in S_1^G(x)$, let T_y be the set of those vertices that are connected to y by a path of length at most r_x not traversing the edge connecting x and y. If $G|_{T_y}$ is not a tree, i.e., $|T_y| < |E(G|_{T_y})| + 1$, or $x \in T_y$, then we include the edge between x and y into $H^{(1)}$. We now show that

$$\max_{x\in\mathcal{V}} D_x^{H^{(1)}} \leqslant \mathcal{C} + q_1 \tag{6.9}$$

with very high probability, where q_i denotes the maximal number of vertices in $\mathcal{V}$ that is in the ball of radius i around a vertex in $\mathcal{V}$, i.e.,

$$q_i := \max_{x\in\mathcal{V}} |\mathcal{V} \cap B_i^G(x) \setminus \{x\}|. \tag{6.10}$$

Let $x \in \mathcal{V}$. Indeed, owing to Lemma 4.5, with very high probability, there are at most $\mathcal{O}(1)$ edges in $B_{r_x}^G(x)$ that prevent it from being a tree. Moreover, $S_1^G(x)$ contains at most q_1 vertices in $\mathcal{V}$. This proves (6.9) and by construction $(G \setminus H^{(1)})|_{B_{r_x/2}^{G\setminus H^{(1)}}(x)}$ is a tree for any $x \in \mathcal{V}$.

Second, the subgraph $H^{(2)} \subset G$ consists of edges incident to vertices $x \in \mathcal{V}$ that are traversed by paths in $G \setminus H^{(1)}$ of length at most $2r_{x,\tau}$ connecting x to another vertex in $\mathcal{V}$. More precisely, for $x \in \mathcal{V}$ we add the following edges to $H^{(2)}$. Since $(G \setminus H^{(1)})|_{B_{r_x/2}^{G\setminus H^{(1)}}(x)}$ is a tree, for each $y \in (\mathcal{V} \cap B_{2r_{x,\tau}}^{G\setminus H^{(1)}}(x)) \setminus \{x\}$, there is a unique vertex $z \in S_1^{G\setminus H^{(1)}}(x)$ such that each path in $G \setminus H^{(1)}$ connecting x and y traverses the edge between x and z. All such edges between x and z are added to $H^{(2)}$. This algorithm yields that

$$\max_{x\in\mathcal{V}} D_x^{H^{(2)}} \leqslant q_{r_\tau}, \tag{6.11}$$

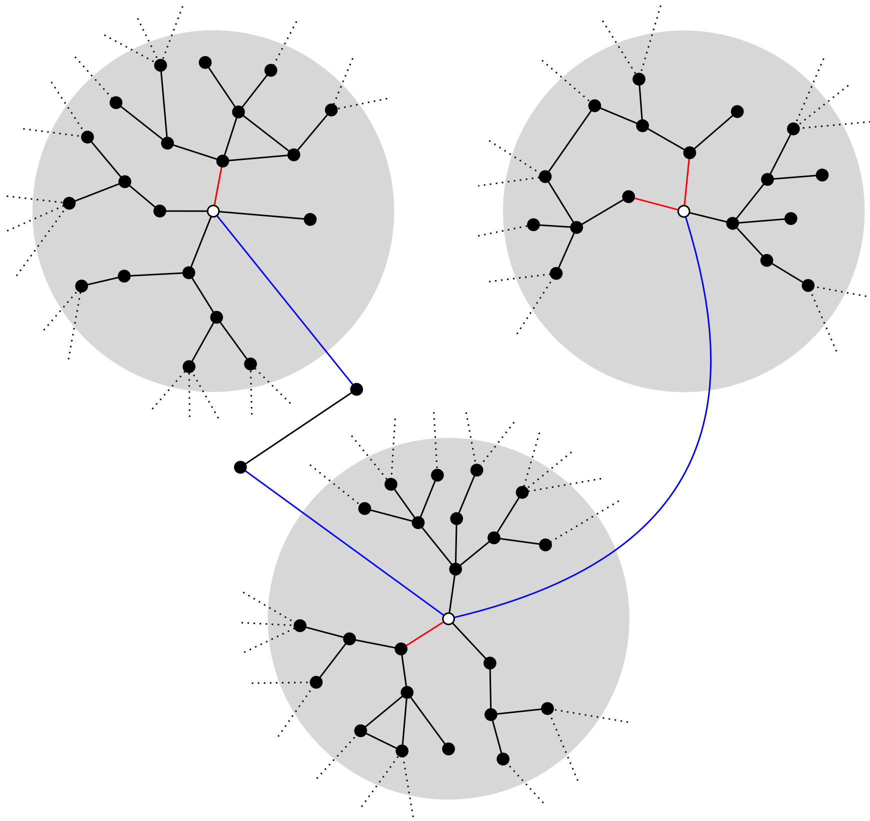

Figure 6.1. A schematic illustration of the algorithm in the proof of Lemma 6.2. The vertices of $\mathcal{V}$ are white and the other vertices black. The balls $B_{r_{x,\tau}}^{G_\tau}(x)$, for each $x \in \mathcal{V}$, are indicated using grey balls, and they are disjoint by construction. The edges of the subgraph $H^{(1)}$ are drawn in red. The edges of the subgraph $H^{(2)}$ are drawn in blue.

where q_{r_τ} is defined in (6.10) with $i = r_\tau$.

We set $H_\tau := H^{(1)} \cup H^{(2)}$ and $G_\tau := G \setminus H_\tau$. By construction, each path in G_τ between $x, y \in \mathcal{V}$ with $x \neq y$ has length at least $2(r_{x,\tau} \vee r_{y,\tau}) + 1$. This establishes property (i) of Lemma 6.2. Moreover, since $G_\tau \subset G \setminus H^{(1)}$ is a subgraph and the latter is a tree when restricted to $B_{r_x}^{G\setminus H^{(1)}}(x)$ we obtain (ii).

For the proof of property (iii) let $x \in \mathcal{V}$ be fixed. The construction of $H^{(1)}$ implies that $S_i^{G\setminus H^{(1)}}(x) \subset S_i^G(x)$ for all $1 \leqslant i \leqslant r_x/2$. As $(G\setminus H^{(1)})|_{B_{r_x/2}^{G\setminus H^{(1)}}(x)}$ is a tree, a vertex lies in $S_i^{G_\tau}(x)$ only if it was in $S_i^{G\setminus H^{(1)}}(x)$ due to the construction of $H^{(2)}$. Hence, $S_i^{G_\tau}(x) \subset S_i^{G\setminus H^{(1)}}(x) \subset S_i^G(x)$ and we deduce (iii). We also note that the construction of H_τ explained above yields

$$E(H_\tau) \subset \bigcup_{x\in\mathcal{V}} \bigcup_{y\in S_1^G(x)} \{x, y\}, \tag{6.12}$$

i.e., for each edge in H_τ, there is at least one vertex $x \in \mathcal{V}$ incident to it. This shows (iv).

For each $x \in [N]$, we now verify the bound on $D_x^{G\setminus G_\tau} = D_x^{H_\tau}$ in (v). For any $x \in \mathcal{V}$, we have

$$D_x^{H_\tau} = D_x^{H^{(1)}} + D_x^{H^{(2)}} \leqslant \mathcal{C} + q_1 + q_{r_\tau} \leqslant \mathcal{C} + 2\max_{y\in\mathcal{V}} |\mathcal{V} \cap B_{r_\tau}(y)|$$

due to (6.9), (6.11) and $q_1 \leqslant q_{r_\tau} \leqslant \max_{y\in\mathcal{V}}|\mathcal{V} \cap B_{r_\tau}(y)|$. Thus, Lemma 6.3 implies (v) for all $x \in \mathcal{V}$. Let $x \in [N] \setminus \mathcal{V}$. If $S_1^G(x) \cap \mathcal{V} = \emptyset$ then $D_x^{H_\tau} = 0$ due to (6.12). If $x \in S_1^G(y)$ for some $y \in \mathcal{V}$ then

$$D_x^{H_\tau} \leqslant \mathcal{C}|S_1^G(x) \cap \mathcal{V}| \leqslant \mathcal{C}|S_2^G(y) \cap \mathcal{V}| \leqslant \mathcal{C}\frac{\log N}{h(\frac{\tau-1}{2})d}$$

by Lemma 6.3. This completes the proof of (v).

What remains is the proof of (6.3). We fix $x \in \mathcal{V}$ and conclude from (6.12) that

$$|S_i^G(x) \setminus S_i^{G_\tau}(x)| \leqslant \sum_{z\in S_1(x)\colon \{x,z\}\in E(H_\tau)} |S_{i-1}^G(z)|.$$

Lemma 6.4 provides a uniform bound on the summands in the previous sum. The number of elements in this sum is bounded by $D_x^{G\setminus G_\tau}$. Hence, we conclude (vi). This completes the proof of Lemma 6.2. □

Proof of Lemma 6.3. The lemma will follow from an estimate on the probability of the event $\Xi^{(k)}$ defined through

$$\Xi^{(k)} := \{\exists\, x \in [N]\colon x \in \mathcal{V}_\tau,\ |\mathcal{V}_\tau \cap B_r(x)| \geqslant k\}$$

for some $k \in \mathbb{N}$ to be chosen later. We decompose $\Xi^{(k)}$ according to

$$\Xi^{(k)} = \bigcup_{x,\mathbf{y},\mathbf{z}} \Xi^{(k)}_{x,\mathbf{y},\mathbf{z}},$$
$$\Xi^{(k)}_{x,\mathbf{y},\mathbf{z}} = \Big\{x, y_j \in \mathcal{V}_\tau,\ \{x, z_1^{(j)}\}, \{z_i^{(j)}, z_{i+1}^{(j)}\}, \{z_{r_j}^{(j)}, y_j\} \in E(G) \text{ for all } i \in [r_j - 1] \text{ and } j \in [k]\Big\}, \tag{6.13}$$

where the union is taken over all $x \in [N]$, k-tuples $\mathbf{y} = (y_1, \dots, y_k)$ of distinct elements of $[N]\setminus\{x\}$ and k-tuples $\mathbf{z} = (z^{(1)}, \dots, z^{(k)})$ of paths $z^{(j)} = (x, z_1^{(j)}, \dots, z_{r_j}^{(j)}, y_j)$ of length $r_j \in \{0, \dots, r\}$ for $j = 1, \dots, k$.

We fix such x, $\mathbf{y} = (y_1, \dots, y_k)$ and $\mathbf{z} = (z^{(1)}, \dots, z^{(k)})$. Then, recalling $p = d/N$, it is easy to see that

$$\mathbb{P}(\Xi^{(k)}_{x,\mathbf{y},\mathbf{z}}) \leqslant P_{x,\mathbf{y}} \prod_{j=1}^k p^{r_j+1}, \tag{6.14}$$

where $P_{x,\mathbf{y}} := \mathbb{P}(D_x \geqslant \tau d - k, D_{y_1} \geqslant \tau d - 1, \dots, D_{y_k} \geqslant \tau d - 1)$. We now show that

$$P_{x,\mathbf{y}} \leqslant C^{k+1} \exp\Big(-d(k+1)h\Big(\frac{\tau-1}{2}\Big)\Big) + \Big(\frac{(k+1)d}{N}\Big)^{d(\tau-1)/2-k}. \tag{6.15}$$

We start the proof of (6.15) by exploiting the fact that $D_x, D_{y_1}, \dots, D_{y_k}$ are almost independent. Indeed, setting $y_0 := x$, $a_0 := (\tau - 1)d - k$, $a_1 := \dots := a_k := (\tau - 1)d - 1$, we obtain

$$P_{x,\mathbf{y}} = \mathbb{E}[\mathbb{P}(D_{y_0} - d \geqslant a_0, \dots, D_{y_k} - d \geqslant a_k \mid A_Y)] = \mathbb{E}\Big[\prod_{i=0}^k \mathbb{P}(D_{y_i} - d \geqslant a_i \mid A_Y)\Big] \tag{6.16}$$

where $Y := \{y_0, \dots, y_k\}$ and in the last step we used that $D_{y_0}, \dots D_{y_k}$ are independent conditionally on A_Y as

$$D_{y_i} - d = X_i - \mathbb{E}[X_i \mid A_Y] + \delta_i, \qquad X_i := \sum_{z\in[N]\setminus Y} A_{zy_i}, \qquad \delta_i := \sum_{z\in Y}\Big(A_{zy_i} - \frac{d}{N}(k+1)\Big) \tag{6.17}$$

and $X_0, \ldots, X_k$ are independent while the remainder is measurable with respect to A_Y. Hence, (6.17) and Bennett's inequality imply

$$\mathbb{P}(D_{y_i} - d \geqslant a_i \mid A_Y) \leqslant C \exp\Big(- dh((a_i - \delta_i)/d)\Big) \leqslant C \exp\Big(- dh\Big(\min_{i=0,\ldots,k} a_i/d - \max_{i=0,\ldots,k} \delta_i/d\Big)\Big).$$

Therefore, since $\min_i a_i = (\tau-1)d - k$, (6.16) yields

$$P_{x,\mathbf{y}} \leqslant C^{k+1} \exp\Big(- d(k+1)h((\tau-1)/2)\Big) + \mathbb{P}\Big(\max_{i=0,\ldots,k} \delta_i > (\tau-1)d/2 - k\Big).$$

We choose $n = (\tau-1)d/2 - k$ and use the bound

$$\mathbb{P}(\delta_i > n) \leqslant \mathbb{P}\Big(\sum_{z \in Y} A_{zy_i} \geqslant n\Big) \leqslant \binom{k+1}{n} \Big(\frac{d}{N}\Big)^n \leqslant \Big(\frac{(k+1)d}{N}\Big)^n$$

to conclude (6.15).

We now finish the proof by combining the previous estimates. In fact, from (6.13), (6.14), $p = d/N$ and (6.15), we conclude

$$\begin{aligned}
\mathbb{P}(\Xi^{(k)}) &\leqslant \sum_{x,\mathbf{y},\mathbf{z}} \mathbb{P}(\Xi^{(k)}_{x,\mathbf{y},\mathbf{z}}) \\
&\leqslant \binom{N}{k+1} \sum_{r_1=0}^{r} \cdots \sum_{r_k=0}^{r} \binom{N-k-1}{r_1} \cdots \binom{N-k-1-\sum_{l=1}^{k-1} r_l}{r_k} p^{k+\sum_{l=1}^k r_l} \max_{x,\mathbf{y}} P_{x,\mathbf{y}} \\
&\leqslant \sum_{r_1,\ldots,r_k=0}^{r} N^{k+1+\sum_{l=1}^k r_l} p^{k+\sum_{l=1}^k r_l} \max_{x,\mathbf{y}} P_{x,\mathbf{y}} \\
&\leqslant N \Big(\sum_{l=0}^{r} d^{l+1}\Big)^k \max_{x,\mathbf{y}} P_{x,\mathbf{y}} \\
&= N \Big(\frac{d^{r+2}-1}{d-1}\Big)^k \Big(C^{k+1} \exp\Big(- d(k+1)h\Big(\frac{\tau-1}{2}\Big)\Big) + \Big(\frac{(k+1)d}{N}\Big)^{d(\tau-1)/2-k}\Big).
\end{aligned}$$

Therefore, in order to obtain (6.8), the condition

$$(k+1)\Big(\log C - dh((\tau-1)/2) + (r+2)\log d\Big) - (r+2)\log d + \log N < -\mathcal{C} \log N$$

has to be satisfied. This condition is met if $k = \mathcal{C} \log N/(dh((\tau-1)/2))$ and $r \leqslant r_\tau$. These choices of k and r also imply that the remaining term in the estimate of $\mathbb{P}(\Xi^{(k)})$ is bounded by $\mathcal{C}N^{-\nu}$. This completes the proof of Lemma 6.3. □

Proof of Lemma 6.4. By Lemma 3.3, $D_z \leqslant \Delta \leqslant \sqrt{N}(2d)^{-r}$ with very high probability if $r \leqslant \log N/(4\log d)$ as $d \leqslant \mathcal{K}^{-1}N^{1/4}$. Hence, a simple induction argument resembling the proof of (4.13b) and using (4.14) and Lemma 3.3 yield

$$|S_i(z)| \leqslant d^{i-1}\Delta\Big[1 + 2\mathcal{C}\sqrt{\frac{\log N}{d\Delta}} \sum_{k=1}^{i-1} d^{-(k-1)/2}\Big] \tag{6.18}$$

with very high probability for all $1 \leqslant i \leqslant \log N/(4\log d)$. Here, we used that $\sqrt{\log N/(d\Delta)}$ is small if $\mathcal{K}$ is large due to $d \geqslant \mathcal{K} \log\log N$, and for $i \geqslant 1$,

$$\sum_{j=1}^{i-1} d^{-(j-1)/2} \leqslant (1 - d^{-1/2})^{-1}. \tag{6.19}$$

Combining (6.18) and (6.19) completes the proof of Lemma 6.4. □

7. Upper bounds on large eigenvalues

The following proposition provides the upper bound on the lth largest eigenvalue matching the lower bound of Proposition 6.1. We recall that the permutation σ was chosen in (2.3).

Proposition 7.1. *Let $\kappa > 0$ be fixed. Abbreviate $a := 1/2 - \kappa$ and suppose that $\kappa \leqslant \theta < 5/2$. Suppose that $d \geqslant (\log N)^{4/(5-2\theta)}$. Define the random index $L_\leqslant$ through*

$$L_\leqslant := \max\{l \geqslant 1 \colon \alpha_{\sigma(l)} \geqslant 2 + (\log d)^{-a}\} \tag{7.1}$$

with the convention that $L_\leqslant = 0$ if $\alpha_x < 2 + (\log d)^{-a}$ for all $x \in [N]$. There is a universal constant $c > 0$ such that the following holds with very high probability.

(i) If $L_\leqslant > 0$ then, for all $l \in [L_\leqslant]$,

$$\max\{\lambda_l(\underline{A}), -\lambda_{N+1-l}(\underline{A})\} \leqslant \sqrt{d}\Lambda(\alpha_{\sigma(l)}) + \mathcal{C}\sqrt{d}\left(d^{-c(\Lambda(\alpha_{\sigma(l)})-2)} + d^{-\theta/2}\right).$$

(ii) If $L_\leqslant = 0$ then

$$\max\{\lambda_1(\underline{A}), \lambda_N(\underline{A})\} \leqslant 2\sqrt{d} + \mathcal{C}\sqrt{d}(\log d)^{-2a}.$$

(Here the constant $\mathcal{C}$ depends on κ.)

Let $L_\leqslant$ be defined as in (7.1). For $l \in [L_\leqslant]$, we set

$$V_l := \begin{cases} [N], & \text{if } l = 1, \\ [N] \setminus \{\sigma(1), \dots, \sigma(l-1)\}, & \text{if } l \geqslant 2. \end{cases} \tag{7.2}$$

Let G_τ be the subgraph of G introduced in Lemma 6.2. We denote by $A_\tau = \mathrm{Adj}(G_\tau)$ the adjacency matrix of G_τ and also define

$$\underline{A}_\tau := A_\tau - \Pi_\tau(\mathbb{E}A)\Pi_\tau, \qquad \mathcal{Z}_\tau := \bigcup_{x \in \mathcal{V}_\tau} B^{G_\tau}_{r_{x,\tau}-2}(x),$$

where Π_τ is the orthogonal projection onto $\mathrm{Span}\{\mathbf{1}_y \colon y \in [N] \setminus \mathcal{Z}_\tau\}$.

For all $\tau, \zeta, \mu > 0$, we define a subset $\mathcal{W}_{\tau,\mu,\zeta}$ of $\mathcal{V}_\tau$ through

$$\mathcal{W}_{\tau,\mu,\zeta} := \{x \in [N] \colon \tau \leqslant \alpha_x,\ \mu \geqslant \sqrt{d}(\Lambda(\alpha_x \vee 2) + \zeta)\}. \tag{7.3}$$

The formulation of the following proposition uses the function $\alpha \colon [2,\infty) \to [2,\infty)$ defined through

$$\alpha(\eta) := \frac{\eta}{2}(\eta + \sqrt{\eta^2 - 4}).$$

Note that α is monotonically increasing and $\Lambda(\alpha(\eta)) = \eta$ for all $\eta \geqslant 2$.

Proposition 7.2 (Delocalization estimate)**.** *Let $d \geqslant (\log N)^{4/(5-2\theta)}$ for some $\theta > 0$ and $1 + d^{-\theta/4} \leqslant \tau \leqslant 2$. Let $L_\leqslant$ be defined as in (7.1), $l \in [L_\leqslant]$ and V_l be defined as in (7.2). Let μ be an eigenvalue of $(\underline{A}_\tau)_{V_l}$ and $\mathbf{w}$ be a corresponding, normalized eigenvector. Then there are a universal constants $c > 0$ and a constant $\mathcal{C} > 0$ such that for any $\zeta > 0$ satisfying*

$$\zeta \geqslant \left(\mathcal{C}\left(\frac{\mu}{\mu - 2\sqrt{d}}\right)^{1/2} d^{-\theta/2}\right) \vee d^{-c(\mu/\sqrt{d}-2)} \tag{7.4}$$

we have

$$\alpha\left(\frac{\mu}{\sqrt{d}}\right) \sum_{x \in \mathcal{W}_{\tau,\mu,\zeta}} \langle \mathbf{w}, \mathbf{1}_x \rangle^2 \leqslant \mathcal{C} d^{-c((\mu/\sqrt{d}-2)\wedge \log(\mu/\sqrt{d}))} \|\mathbf{w}|_{B_r^{G_\tau}(x)}\|^2$$

with very high probability.

Proof of Proposition 7.1. We will only prove the upper bound on the large eigenvalues. The corresponding lower bound on the small eigenvalues is shown similarly (cf. the proof of Proposition 4.1). We first prove (i) assuming $L_{\leqslant} > 0$. We fix $l \in [L_{\leqslant}]$ and set $U_l := \operatorname{span}\{\mathbf{1}_x \colon x \in V_l\}$. The min-max principle implies that

$$\max_{\tilde{\mathbf{w}} \in \mathbb{S}(U_l)} \langle \tilde{\mathbf{w}}, \underline{A}\tilde{\mathbf{w}} \rangle \geqslant \lambda_l(\underline{A}). \tag{7.5}$$

Here and in the following, $\mathbb{S}(U_l)$ denotes the unit sphere with respect to the Euclidean norm in U_l.

By construction, the largest degree of $G|_{V_l}$ is at most $D_{\sigma(l)}$. Let $\tau > 1$ and denote by A_τ the adjacency matrix of G_τ constructed in Lemma 6.2.

Let μ be the largest eigenvalue of $(\underline{A}_\tau)_{V_l}$ and $\mathbf{w}$ be a corresponding normalized eigenvector. Since $\langle \tilde{\mathbf{w}}, (\underline{A}_\tau)_{V_l} \tilde{\mathbf{w}} \rangle = \langle \tilde{\mathbf{w}}, \underline{A}_\tau \tilde{\mathbf{w}} \rangle$ for all $\tilde{\mathbf{w}} \in \mathbb{S}(U_l)$, we get

$$\mu = \max_{\tilde{\mathbf{w}} \in \mathbb{S}(U_l)} \langle \tilde{\mathbf{w}}, (\underline{A}_\tau)_{V_l} \tilde{\mathbf{w}} \rangle = \max_{\tilde{\mathbf{w}} \in \mathbb{S}(U_l)} \langle \tilde{\mathbf{w}}, \underline{A}_\tau \tilde{\mathbf{w}} \rangle. \tag{7.6}$$

Thus, we obtain the lower bound

$$\mu \geqslant -\|\underline{A}_\tau - \underline{A}\| + \max_{\tilde{\mathbf{w}} \in \mathbb{S}(U_l)} \langle \tilde{\mathbf{w}}, \underline{A}\tilde{\mathbf{w}} \rangle. \tag{7.7}$$

On the other hand, (7.6) and Proposition 5.1 imply the upper bound

$$\sqrt{d}\,\langle \mathbf{w}, (I_N + D + E)\mathbf{w} \rangle \geqslant \mu - \|\underline{A}_\tau - \underline{A}\| \geqslant \lambda_l(\underline{A}) - 2\|\underline{A}_\tau - \underline{A}\|. \tag{7.8}$$

Here, we used (7.7) and (7.5) in the last step.

We choose $\tau = 1 + d^{-\theta/4}$. Let $\zeta > 0$ satisfying (7.4). We now assume that

$$\lambda_l(\underline{A}) > \sqrt{d}\Big(\Lambda(\alpha_{\sigma(l)}) + \zeta\Big) + \|\underline{A} - \underline{A}_\tau\|. \tag{7.9}$$

From (7.9), (7.5) and (7.7), we deduce

$$\mu \geqslant \sqrt{d}\Big(\Lambda(\alpha_{\sigma(l)}) + \zeta\Big). \tag{7.10}$$

This implies that $V_l \cap \mathcal{V}_\tau \subset \mathcal{W}_{\tau,\mu,\zeta}$.

Moreover, we conclude that

$$\frac{\mu}{\sqrt{d}} \geqslant \Lambda(\alpha_{\sigma(l)}).$$

We shall apply Proposition 7.2. By contradiction let assume that ζ satisfies the condition (7.4).

Since $\langle \mathbf{w}, \mathbf{1}_x \rangle = 0$ for all $x \in [N] \setminus V_l$ and $[N] \setminus V_l \subset \mathcal{V}_\tau$, we obtain

$$\langle \mathbf{w}, D\mathbf{w} \rangle = \sum_{x \in [N] \setminus \mathcal{V}_\tau} |\langle \mathbf{w}, \mathbf{1}_x \rangle|^2 \alpha_x + \sum_{x \in \mathcal{V}_\tau \cap V_l} |\langle \mathbf{w}, \mathbf{1}_x \rangle|^2 \alpha_x \leqslant \tau + \mathcal{C} d^{-c((\Lambda(\alpha_{\sigma(l)})-2)\wedge 1)} \tag{7.11}$$

with very high probability. Here, the last step follows from Proposition 7.2 and $\alpha_x \leqslant \alpha(\mu/\sqrt{d})$ for all $x \in \mathcal{V}_\tau \cap V_l$.

We use the assumption (7.9) in (7.8), employ (7.11) and obtain

$$1 + \tau + \mathcal{C} d^{-c((\Lambda(\alpha_{\sigma(l)})-2)\wedge 1)} + \|E\| \geqslant \Lambda(\alpha_{\sigma(l)}) + \zeta - d^{-1/2}\|\underline{A}_\tau - \underline{A}\|. \tag{7.12}$$

where we used the last step that

$$\|\underline{A}_\tau - \underline{A}\| \leqslant \|A_\tau - A\| + \|\mathbb{E}(A) - \Pi_\tau(\mathbb{E}A)\Pi_\tau\| \leqslant \max_{x \in [N]} D_x^{G \setminus G_\tau} + 2 = \mathcal{O}\big(1 + d^{1/4-\theta/2}\big)$$

by Lemma 6.2 (v), $h(1+\varepsilon) \geqslant c\varepsilon^2$, $d \geqslant (\log N)^{4/(5-4\theta)}$, $\tau = 1 + d^{-\theta/4}$ and

$$\|\mathbb{E}(A) - \Pi_\tau(\mathbb{E}A)\Pi_\tau\| \leqslant 2 \tag{7.13}$$

with very high probability. For the proof of (7.13), we remark that, by construction of G_τ, all balls $B^{G_\tau}_{r_{x,\tau}}(x)$ for $x \in \mathcal{V}_\tau$ are disjoint and we have

$$|\mathcal{Z}_\tau| = \sum_{x\in\mathcal{V}_\tau} |B^{G_\tau}_{r_{x,\tau}-2}(x)| \leqslant \frac{2}{d^2} \sum_{x\in\mathcal{V}_\tau} |B^{G_\tau}_{r_{x,\tau}}(x)| \leqslant \frac{2N}{d^2}$$

with very high probability, where we use Lemma 4.4 for the middle inequality. Thus, estimating the operator norm by the Hilbert-Schmidt norm yields

$$\|\mathbb{E}(A) - \Pi_\tau(\mathbb{E}A)\Pi_\tau\|^2 \leqslant \frac{d^2}{N^2}\big(|\mathcal{Z}_\tau|^2 + 2|\mathcal{Z}_\tau|(N - |\mathcal{Z}_\tau|)\big) \leqslant \frac{d^2 2|\mathcal{Z}_\tau|N}{N^2} \leqslant \frac{2d^2|\mathcal{Z}_\tau|}{N} \leqslant 4$$

with very high probability. This proves (7.13).

The bound in (7.12) contradicts the choice of l and the definition of $L_\leqslant$ in (7.1) as $\tau > 1$.

$$2 + d^{-\theta/4} + \mathcal{C}d^{-c((\Lambda(\alpha_{\sigma(l)})-2)\wedge 1)} + \|E\| - \Lambda(\alpha_{\sigma(l)}) + d^{-1/2}\|\underline{A}_\tau - \underline{A}\| \geqslant \zeta$$

$$d^{-\theta/4} + \mathcal{C}d^{-c(\Lambda(\alpha_{\sigma(l)})-2)} + \mathcal{C}d^{-1/4-\theta/2} - (\Lambda(\alpha_{\sigma(l)}) - 2) \geqslant \zeta$$

Thus,

$$\mathcal{C}d^{-c(\Lambda(\alpha)-2)} - (\Lambda(\alpha_{\sigma(l)}) - 2) \geqslant \zeta$$

We obtain a contradiction if

$$\Lambda(\alpha_{\sigma(l)}) \geqslant 2 + C(\log d)^{-2a}$$

with any $a \in (0, 1/2)$. This is equivalent to

$$\alpha_{\sigma(l)} \geqslant 2 + C(\log d)^{-a}$$

for any $a \in (0, 1/2)$

We conclude that ζ do not satisfies (7.4) and we obtain (i) of Proposition 7.1.

We now prove (ii). By contradiction we assume that $\mu \geqslant \sqrt{d}(2 + \mathcal{C}\log(d)^{-2a})$ and then $\zeta \geqslant \frac{1}{\sqrt{d}}(\mu - \Lambda(2 + (\log(d)^{-a})) \geqslant \log(d)^{-2a}$ which satisfies (7.4). We follow the same steps as above and obtain

$$\mathcal{C}d^{-c(\log(d)^{-2a})} \geqslant \log(d)^{-2a}$$

which gives a contradiction and completes the proof of Proposition 7.1. □

The condition $\mu \geqslant \sqrt{d}(\Lambda(\alpha_x \vee 2) + \zeta)$ in the definition (7.3) is an upper bound on α_x. In order to makes this upper bound explicit, it is convenient to introduce the parameter $\omega \equiv \omega(\mu, \zeta)$ defined as the unique solution in $(2, \infty)$ of

$$\mu = \sqrt{d}(\Lambda(\omega) + \zeta)\,. \tag{7.14}$$

For the following result, we need the definition

$$r^\omega_\tau := \frac{\log N}{3\log(\omega d)} \wedge \left(\frac{d}{4\log d} h\left(\frac{\tau - 1}{2}\right) - 1\right) - 2\,. \tag{7.15}$$

Note that $r^\omega_\tau \leqslant r_x$ if $x \in [N] \setminus \mathcal{V}_\omega$ and $r^\omega_\tau \leqslant r_\tau$ with r_τ as in (6.2). Hence, owing to Lemma 6.2 (i), the balls $(B^{G_\tau}_r(x))_{x\in\mathcal{V}_\tau\setminus\mathcal{V}_\omega}$ are disjoint for $r = r^\omega_\tau$.

Lemma 7.3. *Let $d \geqslant (\log N)^{4/(5-2\theta)}$ for some $\theta > 0$ and let $1 + d^{-\theta/4} \leqslant \tau \leqslant 2$. Let μ be an eigenvalue of $(\underline{A}_\tau)_{V_l}$ and $\mathbf{w}$ be a corresponding eigenvector. Let ω be the unique solution of (7.14). Then there exist an $r \leqslant r_\tau^\omega$, a universal constants $c > 0$ and a constant $\mathcal{C} > 0$ such that for any $x \in \mathcal{W}_{\tau,\mu,\zeta}$ and $\zeta > 0$ satisfying (7.4) we have*

$$\alpha(\mu/\sqrt{d})\langle \mathbf{w}\,, \mathbf{1}_x\rangle^2 \leqslant \mathcal{C} d^{-c((\mu/\sqrt{d}-2)\wedge \log(\mu/\sqrt{d}))} \|\mathbf{w}|_{B_r^{G_\tau}(x)}\|^2$$

with very high probability.

Proof of Proposition 7.2. Let ω be the unique solution of (7.14) and $\eta := \mu/\sqrt{d}$. Owing to Lemma 7.3 and $r_\tau^\omega \leqslant r_x$ for $\alpha_x \leqslant \omega$, we obtain

$$\alpha(\eta) \sum_{x\in\mathcal{W}_{\tau,\mu,\zeta}} |\langle \mathbf{w}\,, \mathbf{1}_x\rangle|^2 \leqslant \mathcal{C} d^{-c((\eta-2)\wedge\log(\eta))} \sum_{x\in\mathcal{W}_{\tau,\mu,\zeta}} \|\mathbf{w}|_{B^{G_\tau}_{r_\tau^\omega}(x)}\|^2 \leqslant \mathcal{C} d^{-c((\eta-2)\wedge\log(\eta))} \|\mathbf{w}\|^2.$$

Here, we used in the second step that $(B^{G_\tau}_{r_\tau^\omega}(x))_{x\in\mathcal{W}_{\tau,\mu,\zeta}}$ are disjoint sets. As $\mathbf{w}$ is normalized, Proposition 7.2 follows. ☐

The remainder of this subsection is devoted to the proof of Lemma 7.3.

7.1. Proof of Lemma 7.3. For the rest of this section we fix $x \in \mathcal{W}_{\tau,\mu,\zeta}$ and omit it from our notation. For the proof of Lemma 7.3 we shall need some basic facts about the tridiagonalization of matrices, which are summarized in Appendix A, and which we refer to throughout this section. Throughout this section, we only work with indices i of tridiagonal matrices satisfying $i \leqslant m$, where m was defined in Appendix A. This is always a simple application of Lemma 4.5, and we shall not dwell on this issue further.

Proposition 7.4. *Let $\tau \in (1, 2]$. Let $r \leqslant r_\tau^\omega$ and $x \in \mathcal{V}_\tau$. For $k \in \mathbb{N}$ we define the error parameter*

$$\mathcal{E}_{\tau,k} := \frac{(3\sqrt{\tau}+2)^k}{\sqrt{d}} \left[\left(\log(d) + \frac{\log(N)}{d}\right)\left(1 + \frac{\log(N)}{d\tau}\right)\right]^{\frac{1}{2}}. \tag{7.16}$$

Let $\widehat{M} := M^{((\underline{A}_\tau)_{V_l},x)}$ be the tridiagonal matrix associated with $(\underline{A}_\tau)_{V_l}$ around x, and $(\mathbf{g}_k)$ the associated orthogonal basis (see Appendix A). Then there exists a constant $\mathcal{C}$ such that if $\mathcal{E}_{\tau,r} \leqslant \frac{1}{2\mathcal{C}}$ then we have for all $k \leqslant r$ with very high probability

$$\frac{\|\mathbf{g}_k - \mathbf{1}_{S_k^{G_\tau}}\|}{\|\mathbf{1}_{S_k^{G_\tau}}\|} \leqslant \mathcal{C}\mathcal{E}_{\tau,k}, \tag{7.17}$$

and

$$\|\widehat{M}_{[\![r]\!]} - \sqrt{d}M(\alpha_x)\| = \mathcal{O}\left(\sqrt{d}\mathcal{E}_{\tau,r} + \frac{\log N}{\tau d}\left(1 + \frac{\log N}{(\tau-1)^2 d}\right)\right). \tag{7.18}$$

Proof of Lemma 7.3. We denote the standard basis vectors of $\mathbb{R}^{r+1}$ by $\mathbf{e}_0, \dots, \mathbf{e}_r$. Let $\widehat{M} := M^{((\underline{A}_\tau)_{V_l},x)}$ be the tridiagonal matrix associated with $(\underline{A}_\tau)_{V_l}$ around x, and $(\mathbf{g}_k)$ the associated orthogonal basis (see Appendix A). Let $\mathbf{w}$ be an eigenvector of $(\underline{A}_\tau)_{V_l}$ with eigenvalue μ. Then, by the tridiagonal property of $\widehat{M}$, we have $(\widehat{M}_{[\![r]\!]} - \mu I_{r+1})\mathbf{w} \in \operatorname{Span}\{\mathbf{e}_r\}$. Hence, we can apply Proposition C.2 with $\widetilde{M} := \frac{1}{\sqrt{d}}\widehat{M}_{[\![r]\!]}$ to estimate $\frac{w_0^2}{\sum_{i=0}^r w_i^2}$ once we have verified its condition (C.10).

Because $G_\tau|_{B_r^{G_\tau}}$ is a tree, $\widetilde{M}_{00} = \widetilde{M}_{11} = 0$ by Lemma A.2 and $\widetilde{M}_{01} = \sqrt{\alpha_x}$. From (7.18) with $r = 2$, we conclude

$$|1 - \widetilde{M}_{12}| \leqslant \varepsilon_2 := \mathcal{C} d^{-\theta/2}. \tag{7.19}$$

Throughout this proof, we need some Lipschitz-type bounds on $\Lambda(\alpha)$ and $\alpha(\eta)$. We have

$$\eta - \Lambda(\alpha \vee 2) \leqslant C\Big(\frac{\alpha(\eta) - 2}{\alpha(\eta)^{3/2}}\Big)(\alpha(\eta) - \alpha \vee 2) \tag{7.20}$$

for all $\alpha > 0$ and $\eta > \Lambda(\alpha \vee 2)$. The proof of (7.20) is a consequence of

$$\Lambda(\alpha(\eta)) - \Lambda(\alpha \vee 2) = \int_{\alpha \vee 2}^{\alpha(\eta)} \Lambda'(t)\mathrm{d}t = \int_{\alpha \vee 2}^{\alpha(\eta)} \frac{t-2}{2(t-1)^{3/2}}\mathrm{d}t$$

and distinguishing cases $\alpha > \alpha(\eta)/2$ and $\alpha \leqslant \alpha(\eta)/2$.

With the notation $\eta = \frac{\mu}{\sqrt{d}}$, we calculate δ introduced in (C.8) below. From (7.19) we conclude

$$|\delta(0, \alpha_x, 0, \widetilde{M}_{12}, \eta)| = \frac{|\gamma(\eta)\eta^2 - \gamma(\eta)\alpha_x - \eta\widetilde{M}_{12}|}{|\alpha_x + \gamma(\eta)\eta\widetilde{M}_{12} - \eta^2|} \leqslant \frac{C\eta}{|\alpha_x - \alpha(\eta)|}$$

for some universal constant $C > 0$. Here, we used that the first condition on ζ in (7.4) together with (7.20) implies $|\alpha_x - \alpha(\eta)| \geqslant \varepsilon_2/4$.

We now verify that the choice

$$r = c\log(\zeta d^{1/4+\theta/2}\sqrt{(\eta-2)/\eta}) + \tilde{\mathcal{C}}$$

for some universal constant $c > 0$ and some $\tilde{\mathcal{C}} > 0$ implies

$$\eta^2 \geqslant 4 + \frac{C(1+\eta)^2\varepsilon_r^2}{(1-\gamma(\eta)^2)^2}(1 + 1 \vee \delta^2), \qquad \varepsilon_r := \mathcal{C}\Big(\mathcal{E}_{\tau,r} + \frac{\log N}{\tau d}\Big(1 + \frac{\log N}{(\tau-1)^2 d}\Big)\Big). \tag{7.21}$$

The bound in (7.21) is equivalent to

$$\varepsilon_r^2 \leqslant \frac{(\eta^2-4)(1-\gamma(\eta)^2)^2}{C\eta^2(1 \vee \delta^2)}. \tag{7.22}$$

$$\varepsilon_r \leqslant 8^r d^{-1/4-\theta/2} + \frac{d^{-\theta}}{(\tau-1)^2} \leqslant 8^r d^{-1/4-\theta/2} + d^{-\theta/2}.$$

By the definition of r and the first condition on ζ in (7.4), we get

$$\begin{aligned} d^{-1/4-\theta/2}\mathrm{e}^{r/c} + d^{-\theta/2} &\leqslant \mathcal{C}\sqrt{\frac{\eta-2}{\eta}}\,\zeta \leqslant \mathcal{C}\sqrt{\frac{\eta-2}{\eta}}\frac{(\alpha(\eta)-2)(\alpha(\eta)-\alpha_x)}{\eta^3} \leqslant \mathcal{C}\sqrt{\frac{\eta-2}{\eta}}\frac{\alpha(\eta)-2}{\eta^2|\delta|} \\ &\leqslant \mathcal{C}\sqrt{\frac{\eta^2-4}{\eta^2}}\frac{\alpha(\eta)-2}{\eta|\delta|} \leqslant \mathcal{C}\sqrt{\eta^2-4}\frac{1-\gamma^2}{\eta|\delta|} \end{aligned}$$

if c is chosen sufficiently small.

This proves (7.21) due to (7.22).

Hence, we have justified the conditions of Proposition C.2. It implies

$$\begin{aligned} \frac{w_0^2}{\sum_{i=0}^r w_i^2} &\leqslant \frac{C\alpha_x}{(\alpha_x - \alpha(\eta) + O(\eta^2\varepsilon_2))^2}(\gamma_{\geqslant})^{-2r} \\ &\leqslant \frac{\alpha_x}{(1+\eta)^4}\frac{C}{\zeta^2}\Big(1 + \frac{1}{4}(\eta-2)\Big)^{-2(r-2)} \\ &\leqslant \frac{C}{\alpha(\eta)\zeta^2}\Big(1 + \frac{1}{4}(\eta-2)\Big)^{-2(r-2)} \end{aligned}$$

where we used (C.11) in the last step.

Thus, our choice of r, the first condition in (7.4) and the definition of η yield

$$\begin{aligned}\alpha(\eta)\frac{w_0^2}{\sum_{i=0}^r w_i^2} &\leqslant \frac{C}{\zeta^2}\exp\Big(-2(r-2)\log\Big(1+\frac{1}{4}(\eta-2)\Big)\Big)\\ &\leqslant \frac{\mathcal{C}}{\zeta^2}\exp\Big(-\frac{c}{2}\log d\log\Big(1+\frac{1}{4}(\eta-2)\Big)\Big)\\ &\leqslant \mathcal{C}\exp\Big(-c\log d\big((\eta-2)\wedge\log\eta\big)\Big)\end{aligned}$$

Here, we used the second condition on ζ in (7.4) in the third step.

To conclude the proof, we note that

$$\frac{w_0^2}{\sum_{i=0}^r w_i^2} \geqslant \frac{|w_0|^2}{\|\mathbf{w}|_{B_r^{G_\tau}}\|^2}, \tag{7.23}$$

where we used

$$\|\mathbf{w}|_{B_r^{G_\tau}}\|^2 \geqslant \sum_{i=0}^r \Big|\Big\langle \frac{\mathbf{g}_i}{\|\mathbf{g}_i\|}\ ,\ \mathbf{w}\Big\rangle\Big|^2 = \sum_{i=0}^r |w_i|^2.$$

because $\mathbf{g}_i \in \operatorname{Span}\{\mathbf{1}_y : y \in B_r^{G_\tau}\}$ for all $i \leqslant r$ and are orthogonal. ☐

7.2. Proof of Proposition 7.4.

Proof of Proposition 7.4. We first remark that A_τ and $(\underline{A}_\tau)_{V_l}$ agree in the vicinity of $x \in \mathcal{V}_\tau \cap V_l$ in the sense that

$$((\underline{A}_\tau)_{V_l})^i\mathbf{1}_x = (A_\tau)^i\mathbf{1}_x$$

for all $i \in [\![r_{x,\tau}-2]\!]$.

For the proof of (7.17), we now introduce a second family $(\mathbf{f}_k)_k$ of vectors that will turn out to be a good approximation of $(\mathbf{g}_k)_k$. The vectors $\mathbf{f}_k$ are defined through

$$\mathbf{f}_0 = \mathbf{1}_x, \qquad \mathbf{f}_1 = \mathbf{1}_{S_1^{G_\tau}}, \qquad \mathbf{f}_2 = \mathbf{1}_{S_2^{G_\tau}}, \qquad \mathbf{f}_{k+2} = Q_0\big(A_\tau\mathbf{f}_{k+1} - d\mathbf{f}_k\big)$$

for all $k \geqslant 1$. Here and in the following, Q_i denotes the orthogonal projection on $\operatorname{Span}\{(A_\tau)^j\mathbf{1}_x : j \in [\![i]\!]\}^\perp$ as in Appendix A.

The careful analysis of $\mathbf{f}_k$ presented below will imply (7.17) due to the bound

$$\|\mathbf{g}_k - \mathbf{1}_{S_k^{G_\tau}}\| \leqslant \|\mathbf{q}_k\|, \tag{7.24}$$

where we introduced

$$\mathbf{q}_k := \mathbf{f}_k - \mathbf{1}_{S_k^{G_\tau}}.$$

Before estimating $\mathbf{q}_k$, we now establish (7.24). It is easy to check that there exists a monic polynomial P_k of degree k such that $f_k = P_k(A_\tau)1_x$ and then

$$\mathbf{g}_k = Q_{k-1}(A_\tau^k 1_x) = Q_{k-1}(P_k(A_\tau)1_x) = Q_k\mathbf{f}_k.$$

Hence, $\|\mathbf{g}_k\| \leqslant \|\mathbf{f}_k\|$ and, thus, we have

$$\|\mathbf{g}_k - \mathbf{1}_{S_k^{G_\tau}}\|^2 = \|\mathbf{g}_k\|^2 - \|\mathbf{1}_{S_k^{G_\tau}}\|^2 \leqslant \|\mathbf{f}_k\|^2 - \|\mathbf{1}_{S_k^{G_\tau}}\|^2 = \|\mathbf{q}_k\|^2.$$

Here, we used in the first step that $\mathbf{g}_k - \mathbf{1}_{S_k^{G_\tau}}$ is orthogonal to $\mathbf{1}_{S_k^{G_\tau}}$. This is a consequence of $\mathrm{supp}(\mathbf{g}_k - \mathbf{1}_{S_k^{G_\tau}}) \subset B_{k-2}^{G_\tau}$ which follows from the fact that $G_\tau|_{B^{G_\tau}_{r_{x,\tau}}}$ is a tree. In the last step, we used $\mathbf{q}_k \perp \mathbf{1}_{S_k^{G_\tau}}$ which can be shown by a similar argument. This shows (7.24).

Owing to (7.24), the bound in (7.17) follows directly from

$$\|\mathbf{q}_k\| \leqslant \tilde{c}_k \|\mathbf{1}_{S_k^{G_\tau}}\| \tag{7.25}$$

which holds with very high probability for all $k \leqslant r_{x,\tau} - 2$ as we shall show below. Here, $\tilde{c}_k$ is the unique solution of

$$\tilde{c}_{k+2} = 2\tilde{c}_k + 3\sqrt{\tau}\tilde{c}_{k+1} + \frac{\mathcal{C}}{\sqrt{d}}\Big([(\log(d) + \frac{\log(N)}{d})(1 + \frac{\log(N)}{D_x})]^{\frac{1}{2}}\Big)$$

with the initial choices $\tilde{c}_1 = 0$ and $\tilde{c}_2 = 0$.

We now prove (7.25) by induction on k. The induction beginning for $k = 1$ and $k = 2$ is trivial. For the induction step, we decompose

$$\begin{aligned} A_\tau \mathbf{f}_{k+1} - d\mathbf{f}_k &= A_\tau(\mathbf{1}_{S_{k+1}^{G_\tau}} + \mathbf{q}_{k+1}) - d(\mathbf{1}_{S_k^{G_\tau}} + \mathbf{q}_k) \\ &= \mathbf{1}_{S_{k+2}^{G_\tau}} + \sum_{y \in S_k^{G_\tau}} (N_{k+1}(y) - d)\mathbf{1}_y + A_\tau \mathbf{q}_{k+1} - d\mathbf{q}_k \\ &= \mathbf{1}_{S_{k+2}^{G_\tau}} + \sum_{y \in S_k^{G_\tau}} (N_{k+1}(y) - \frac{|S_{k+1}|}{|S_k|})\mathbf{1}_y + (\frac{|S_{k+1}|}{|S_k|} - d)\mathbf{1}_{S_k^{G_\tau}} + A_\tau \mathbf{q}_{k+1} - d\mathbf{q}_k. \end{aligned}$$

Here, we used in the second step that $\langle \mathbf{1}_y \,, A\mathbf{1}_{S_{k+1}^{G_\tau}} \rangle = N_{k+1}(y)$ as $A_\tau \mathbf{1}_{S_k^{G_\tau}} = A\mathbf{1}_{S_k}$ for all $k > 1$. This relation follows from Lemma 6.2 (iv). Therefore, following the proof of (4.10c) yields

$$\begin{aligned} \frac{1}{\sqrt{|S_{k+1}|}} \| \sum_{y \in S_k^{G_\tau}} (N_{k+1}(y) - \frac{|S_{k+1}|}{|S_k|})\mathbf{1}_y \| &\leqslant \frac{1}{\sqrt{|S_{k+1}|}} \| \sum_{y \in S_k} (N_{k+1}(y) - \frac{|S_{k+1}|}{|S_k|}) 1_y \| \\ &= \mathcal{O}\Big([(\log(d) + \frac{\log(N)}{d})(1 + \frac{\log(N)}{D_x})]^{\frac{1}{2}}\Big) \end{aligned}$$

with very high probability. Moreover using Lemma 4.4 we have

$$\frac{1}{\sqrt{|S_{k+1}|}} \|(\frac{|S_{k+1}|}{|S_k|} - d)\mathbf{1}_{S_k^{G_\tau}}\| \leqslant \frac{\sqrt{|S_k|}}{\sqrt{|S_{k+1}|}} \|(\frac{|S_{k+1}|}{|S_k|} - d)\| \leqslant \mathcal{O}([\frac{\log(N)}{dD_x}]^{\frac{1}{2}})$$

which is very small compare to the estimate just above. Since A_τ has degree at most τd on $B_r^{G_\tau} \setminus \{x\}$ then $\|A_\tau v\| \leqslant 2\sqrt{\tau d}\|v\|$ for all $\mathbf{v}$ with $\mathrm{supp}(\mathbf{v}) \subset B_r^{G_\tau}$ and $\langle \mathbf{v} \,, \mathbf{1}_x \rangle = 0$ [15, Chap. 11, Ex. 14]. Therefore we have

$$\|A_\tau \mathbf{q}_{k+1}\| = \|A_\tau Q_0 \mathbf{q}_{k+1}\| \leqslant 2\sqrt{\tau d}\|\mathbf{q}_{k+1}\|.$$

We put everything together and we get

$$\begin{aligned} \|\mathbf{q}_{k+2}\| &= \|Q_0(\mathbf{f}_{k+2} - \mathbf{1}_{S_{k+2}^{G_\tau}})\| \\ &\leqslant d\|\mathbf{q}_k\| + 2\sqrt{\tau d}\|\mathbf{q}_{k+1}\| + \sqrt{|S_{k+1}|}\mathcal{O}\Big([(\log(d) + \frac{\log(N)}{d})(1 + \frac{\log(N)}{D_x})]^{\frac{1}{2}}\Big). \end{aligned}$$

We set $c_k = \frac{\|\mathbf{q}_k\|}{\sqrt{|S_k|}}$. Thus, the previous estimate implies

$$c_{k+2} \leqslant \frac{d\sqrt{|S_k|}}{\sqrt{|S_{k+2}|}}c_k + \frac{2\sqrt{\tau d|S_{k+1}|}}{\sqrt{|S_{k+2}|}}c_{k+1} + \frac{\sqrt{|S_{k+1}|}}{\sqrt{|S_{k+2}|}}\mathcal{O}\Big([(\log(d) + \frac{\log(N)}{d})(1 + \frac{\log(N)}{D_x})]^{\frac{1}{2}}\Big).$$

We use Lemma 4.4 and obtain

$$c_{k+2} \leqslant (1+o(1))c_k + 2\sqrt{\tau}(1+o(1))c_{k+1} + \frac{1}{\sqrt{d}}\mathcal{O}\Big([(\log(d) + \frac{\log(N)}{d})(1 + \frac{\log(N)}{D_x})]^{\frac{1}{2}}\Big)$$

This completes the induction step and, thus, the proof of (7.25).

We now verify (7.18). We have

$$\|g_i\|^2 = |S_i|\Big(1 + (\frac{|S_i^{G_\tau}|}{|S_i|} - 1)\Big)\Big(1 + (\frac{\|g_i\|^2}{|S_i^{G_\tau}|} - 1)\Big) = |S_i|\Big(1 + \mathcal{O}\Big(\mathcal{E}_{\tau,r} + \frac{\log N}{\tau\sqrt{d}}\Big(1 + \frac{\log N}{(\tau-1)^2 d}\Big)\Big)\Big)$$

To estimate the last term, we use (6.3), Lemma 6.2 (iii) and $\mathcal{K}\log N \leqslant D_x d$. By Lemma A.2, we have $\widetilde{M}_{ii} = 0$ and

$$\widetilde{M}_{i\ i+1} = \frac{\|\mathbf{g}_{i+1}\|}{\|\mathbf{g}_i\|} = \sqrt{\frac{|S_{i+1}|}{|S_i|}}\Big(1 + \mathcal{O}\Big(\mathcal{E}_{\tau,r} + \frac{\log N}{\tau\sqrt{d}}\Big(1 + \frac{\log N}{(\tau-1)^2 d}\Big)\Big)\Big).$$

and we conclude using Lemma 4.4 □

8. Proofs of the results in Section 2

In this short section we state how to conclude the results of Section 2. Theorem 2.1 follows from Propositions 6.1 and 7.1, noting that $L = L_{\leqslant} \leqslant L_{\geqslant}$. Corollary 2.3 follows from Theorem 2.1 by eigenvalue interlacing, $\lambda_l(\underline{A}) \geqslant \lambda_{l+1}(A) \geqslant \lambda_{l+1}(\underline{A})$ for $1 \leqslant l \leqslant N-1$, as well as the mean value theorem. Finally, the proof of Theorem 2.5 is very similar to that of Theorem 2.1, and we explain the needed minor modifications in the next section.

9. Modifications for sparse Wigner matrices

In this section, we explain how the arguments in the previous sections can be adapted to yield the proof of Theorem 2.5. We consider a sparse Wigner matrix X with entries $X_{xy} = W_{xy}A_{xy}$. Here, A is the adjacency matrix of an Erdős-Rényi graph on $[N]$ with edge probability d/N and $W = (W_{xy})_{x,y\in[N]}$ is an independent Wigner matrix with bounded entries. That is, W is Hermitian and the random variables $(W_{xy}\colon x \leqslant y)$ are independent and

$$\mathbb{E}W_{xy} = 0, \qquad \mathbb{E}|W_{xy}|^2 = 1, \qquad |W_{xy}| \leqslant K \tag{9.1}$$

for all $x, y \in [N]$ and some constant $K > 0$.

The assumptions imply that X is symmetric and we consider X as the adjacency matrix of an undirected weighted graph with edge weights W_{xy}. According to this philosophy, we define

$$S_i(x) := \{y \in [N]\colon \min\{j \geqslant 0\colon (X^j)_{xy} \neq 0\} = i\}, \qquad B_i(x) := \bigcup_{j\in[\![i]\!]} S_j(x)$$

for all $x \in [N]$.

In the remainder of this section, we explain the necessary adjustments in order to conclude Theorem 2.5 along the proof of Theorem 2.1 with the definition of α_x from (2.6) and $D_x = \alpha_x d$. Throughout this section, the constant $\mathcal{C}$ as well as the implicit constant in $\mathcal{O}$ are also allowed to depend on K, the uniform bound on W_{xy} in (9.1). With this convention, the arguments in Section 5 to Section 7 do not require any changes. They only have to be understood with respect to the new definition of $\mathcal{C}$ and $\mathcal{O}$. The necessary modifications of Section 4 are explained in the following subsection. Once they are taken into account Theorem 2.5 follows from Propositions 6.1 and 5.1.

9.1. Modifications in Section 4. In this subsection, we fix $x \in [N]$ and explain the modifications required in Section 4 to obtain the corresponding results in the setup described above.

Definition of the approximate eigenvector, decomposition of the error terms. We now introduce the analogue of the approximate eigenvector $\mathbf{b}$ from (4.3) in the present setup. We define $\mathbf{g}_0 := \mathbf{1}_x$. For $i \geqslant 1$, we define

$$\mathbf{g}_i := (X\mathbf{g}_{i-1})|_{S_i(x)}.$$

Note that $\mathbf{g}_0, \ldots, \mathbf{g}_i$ is an orthogonal basis of $\operatorname{Span}\{X^j \mathbf{1}_x : j \in [\![i]\!]\}$.

With the choices of u_i from (4.2), we set

$$\mathbf{v} = \sum_{i=0}^{r} u_i \hat{\mathbf{g}}_i, \qquad \hat{\mathbf{g}}_i := \frac{\mathbf{g}_i}{\|\mathbf{g}_i\|}.$$

Similarly to the proof of Lemma 4.2, we obtain

$$(X - \sqrt{d}\Lambda(\alpha_x))\mathbf{v} = \mathbf{w}_1 + \ldots + \mathbf{w}_4,$$

where the error terms $\mathbf{w}_1, \ldots, \mathbf{w}_4$ are defined through

$$\begin{aligned}
\mathbf{w}_1 &:= \sum_{i=0}^{r} \frac{u_i}{\|\mathbf{g}_i\|}\bigg(\sum_{y \in S_{i+1}} \Big(N_i(y) - \langle \mathbf{1}_y \,, \mathbf{g}_{i+1} \rangle \Big) \mathbf{1}_y + \sum_{y \in S_i} N_i(y) \mathbf{1}_y \bigg), \\
\mathbf{w}_2 &:= \sum_{i=1}^{r} \frac{u_i}{\|\mathbf{g}_i\|}\bigg(\sum_{y \in S_{i-1}} N_i(y)\mathbf{1}_y - \frac{\|\mathbf{g}_i\|^2}{\|\mathbf{g}_{i-1}\|^2}\mathbf{g}_{i-1} \bigg), \\
\mathbf{w}_3 &:= u_2 \left(\frac{\|\mathbf{g}_2\|}{\|\mathbf{g}_1\|} - \sqrt{d} \right) \hat{\mathbf{g}}_1 + \sum_{i=2}^{r-1} \left[u_{i+1} \left(\frac{\|\mathbf{g}_{i+1}\|}{\|\mathbf{g}_i\|} - \sqrt{d} \right) + u_{i-1} \left(\frac{\|\mathbf{g}_i\|}{\|\mathbf{g}_{i-1}\|} - \sqrt{d} \right) \right] \hat{\mathbf{g}}_i, \\
\mathbf{w}_4 &:= \left(u_{r-1} \frac{\|\mathbf{g}_r\|}{\|\mathbf{g}_{r-1}\|} - u_{r-1}\sqrt{d} - u_{r+1}\sqrt{d} \right) \hat{\mathbf{g}}_r + u_r \frac{\|\mathbf{g}_{r+1}\|}{\|\mathbf{g}_r\|} \hat{\mathbf{g}}_{r+1} \,.
\end{aligned}$$

Here, $N_i(y) := \langle \mathbf{1}_y \,, X\mathbf{g}_i \rangle$ for all $y \in [N]$. We remark that the analogue of $\mathbf{w}_0$ vanishes as the entries W_{yz} are centred for all $y, z \in [N]$.

Concentration of $\|\mathbf{g}_{i+1}\|/\|\mathbf{g}_i\|$. In order to establish that the ratio $\frac{\|\mathbf{g}_{i+1}\|}{\|\mathbf{g}_i\|}$ concentrates around $\sqrt{d}$ if $i \geqslant 1$ we follow the proof of Lemma 4.4. It suffices to verify (4.16) in the new setup. Using Bennett's inequality, it is easy to see that

$$\left| \frac{\|\mathbf{g}_{i+1}\|_p^p}{d\|\mathbf{g}_i\|_p^p} - 1 \right| \leqslant \mathcal{C}(1 + K^p) \sqrt{\frac{\log N}{d}} \frac{\|\mathbf{g}_i\|_{2p}^p}{\|\mathbf{g}_i\|_p^p} \tag{9.2}$$

with very high probability for all $p \geqslant 2$. Here, K is the uniform upper bound on W_{xy}.

We choose $p = N$ in (9.2), use that $\|\cdot\|_N$, $\|\cdot\|_{2N}$ and $\|\cdot\|_\infty$ are comparable and obtain

$$\|\mathbf{g}_i\|_\infty \leqslant \mathcal{C}(1+K)^i \tag{9.3}$$

with very high probability. Hence, we obtain $\|\mathbf{g}_i\|_4^2 \leqslant \mathcal{C}(1+K)^i\|\mathbf{g}_i\|_2$ with very high probability. Thus, choosing $p = 2$ in (9.2) yields the desired analogue of (4.16) in the setup of sparse Wigner matrices.

Estimate on $\mathbf{w}_1$. We remark that the $i = 0$ contribution in the definition of $\mathbf{w}_1$ vanishes as $N_0(y) = \langle \mathbf{1}_y\,, X\mathbf{g}_0\rangle = \langle \mathbf{1}_y\,, \mathbf{g}_1\rangle$ for any y as $A_{xx} = 0$. Moreover, $A_{xx} = 0$ also implies that $N_0(y) = 0$ for $y \in S_0$. Hence,

$$\|\mathbf{w}_1\|^2 = \Bigg\|\sum_{i=1}^r \frac{u_i}{\|\mathbf{g}_i\|}\sum_{y\in S_i} N_i(y)\mathbf{1}_y\Bigg\|^2 = \sum_{i=1}^r \frac{u_i^2}{\|\mathbf{g}_i\|^2}\sum_{y\in S_i} N_i(y)^2.$$

Here, we also used that $N_i(y) - \langle \mathbf{1}_y\,, \mathbf{g}_{i+1}\rangle = \langle \mathbf{1}_y\,, (X\mathbf{g}_i)|_{[N]\setminus S_{i+1}(x)}\rangle = 0$ for any $y \in S_{i+1}$ due to the fact that $(X\mathbf{g}_i)|_{[N]\setminus S_{i+1}(x)}$ vanishes on $S_{i+1}(x)$.

Thus, in order to estimate $\|\mathbf{w}_1\|^2$, we use the following version of (4.22) in Corollary 4.7. Namely, for all $i \geqslant 1$, the bounds

$$\sum_{y\in S_i(x)} N_i(y)^2 = \sum_{y\in S_i(x)} \langle \mathbf{1}_y\,, X\mathbf{g}_i\rangle^2 = \sum_{y\in S_i(x)}\Bigg(\sum_{y_1\in S_i(x)}\langle X\mathbf{1}_y\,, \mathbf{1}_{y_1}\rangle\langle \mathbf{1}_{y_1}\,, \mathbf{g}_i\rangle\Bigg)^2 \leqslant \mathcal{C}\|\mathbf{g}_i\|_\infty^2$$

hold with very high probability. In the last step, we used Lemma 4.5 to conclude that there are at most $\mathcal{O}(1)$ many nonzero terms. Therefore, (9.3) yields (4.10b) in the current setup due to the growth of $\|\mathbf{g}_i\|$.

Estimate on $\mathbf{w}_2$. Here, we follow the proof of (4.10c). By the Pythagorean theorem, we have

$$\|\mathbf{w}_2\|^2 = \sum_{i=1}^{r-1}\frac{u_{i+1}^2}{\|\mathbf{g}_{i+1}\|^2}\sum_{y\in S_i}\bigg(N_{i+1}(y) - \frac{\|\mathbf{g}_{i+1}\|^2}{\|\mathbf{g}_i\|^2}\langle \mathbf{1}_y\,, \mathbf{g}_i\rangle\bigg)^2,$$

where we used that $S_0 = \{x\}$, $N_1(x) = \langle X\mathbf{1}_x\,, \mathbf{g}_1\rangle = \|\mathbf{g}_1\|^2$ as $\mathbf{g}_1 = X\mathbf{1}_x$ due to $A_{xx} = 0$. As $\sum_{i=1}^{r-1} u_{i+1}^2 \leqslant 1$, we obtain

$$\|\mathbf{w}_2\|^2 \leqslant \max_{i\in[r-1]}\frac{1}{\|\mathbf{g}_{i+1}\|^2}\sum_{y\in S_i}\bigg(N_{i+1}(y) - \frac{\|\mathbf{g}_{i+1}\|^2}{\|\mathbf{g}_i\|^2}\langle \mathbf{1}_y\,, \mathbf{g}_i\rangle\bigg)^2 \leqslant \frac{4}{d}\max_{i\in[r-1]}(Z_i + \tilde{Y}_i)$$

where we introduced

$$Z_i := \frac{1}{\|\mathbf{g}_i\|^2}\sum_{y\in S_i}\Big(N_{i+1}(y) - \mathbb{E}[N_{i+1}(y)|X_{(B_{i-1})}]\Big)^2,$$

$$\tilde{Y}_i := \frac{1}{\|\mathbf{g}_i\|^2}\sum_{y\in S_i}\bigg(\mathbb{E}[N_{i+1}(y)|X_{(B_{i-1})}] - \frac{\|\mathbf{g}_{i+1}\|^2}{\|\mathbf{g}_i\|^2}\langle \mathbf{1}_y\,, \mathbf{g}_i\rangle\bigg)^2.$$

We first estimate $\tilde{Y}_i$. As $\mathbb{E}[N_{i+1}(y)|X_{(B_{i-1})}] = \langle \mathbf{1}_y\,, \mathbf{g}_i\rangle d(1 - \frac{|B_i|}{N})$, we conclude

$$\tilde{Y}_i = \frac{1}{\|\mathbf{g}_i\|^2}\bigg(d - \frac{\|\mathbf{g}_{i+1}\|^2}{\|\mathbf{g}_i\|^2} - \frac{d|B_i|}{N}\bigg)^2\sum_{y\in S_i}\langle \mathbf{1}_y\,, \mathbf{g}_i\rangle^2 \leqslant 2\bigg(\bigg(d - \frac{\|\mathbf{g}_{i+1}\|^2}{\|\mathbf{g}_i\|^2}\bigg)^2 + 1\bigg) \leqslant \mathcal{C}\bigg(\frac{d\log N}{D_x} + 1\bigg)$$

with very high probability for all $i \in [r-1]$.

In order to estimate Z_i, we follow the proof of (4.25) and explain the necessary changes. We redefine

$$E_y := \frac{1}{\langle \mathbf{1}_y \,, \mathbf{g}_i \rangle} \Big(N_{i+1}(y) - \mathbb{E}[N_{i+1}(y) | X_{(B_{i-1})}] \Big).$$

and use Bennett's inequality to obtain

$$\mathbb{P}\Big(E_y^2 > (s\kappa d)^2 \;\Big|\; X_{(B_{i-1})} \Big) \leqslant \exp\big(- c\kappa d (s \wedge s^2) \big),$$

where $\kappa = \mathbb{E} W_{yz}^4$ for some $y, z \in [N]$. Hence, using this bound in the proof of (4.27) yields

$$\mathbb{P}(L_s^i \geqslant \ell \mid X_{(B_{i-1})}) \leqslant \binom{\|\mathbf{g}_i\|^2}{\ell} \exp\big(- c\kappa d \ell (s \wedge s^2) \big).$$

Applying this estimate in the remainder of the proof of (4.25), we deduce

$$|Z_i| \leqslant \mathcal{C}(1+K)^{2i} d \Big(1 + \frac{\log N}{\|\mathbf{g}_i\|^2} \Big) \Big(\log d + \frac{\log N}{d} \Big).$$

Here, we employed that $\|\mathbf{g}_i\|_\infty \leqslant \mathcal{C}(1+K)^i$. Therefore, using the growth of $\|\mathbf{g}_i\|$, we obtain the same bound on $\|\mathbf{w}_2\|$ as in (4.10c).

When following the proof of (4.10c) in the proof of Proposition 7.4, the same adjustments yield the bound used there.

A. Tridiagonalization

Let $X \in \mathbb{R}^{N \times N}$ be a symmetric matrix and $x \in [N]$. Let $m(x) := \dim \operatorname{Span}\{X^n \mathbf{1}_x : n \in \mathbb{N}_0\}$. For $i \in [\![m]\!]$ define by induction

$$\mathbf{g}_0 := \mathbf{1}_x \,, \qquad \mathbf{g}_{i+1} := Q_i X \mathbf{g}_i \,,$$

where Q_i is the orthogonal projection onto the orthogonal complement of $\operatorname{Span}\{X^j \mathbf{1}_x \colon j \in [\![i]\!]\}$. We call $(\mathbf{g}_i)_{i \in [\![m-1]\!]}$ the orthogonal basis associated with X and x. Note that this basis is in general not normalized. For convenience, if $m < N-1$, i.e. $\mathbf{1}_x$ is not a cyclic vector of X, we complete the basis $(\mathbf{g}_i)_{i \in [\![m-1]\!]}$ to an orthogonal basis $(\mathbf{g}_i)_{i \in [\![N-1]\!]}$ of $\mathbb{R}^N$ in an arbitrary fashion.

We define $M \equiv M^{(X,x)}$ as the matrix X in the orthonormal basis $(\mathbf{g}_i / \|\mathbf{g}_i\|)_{i \in [\![N-1]\!]}$; that is,

$$M_{ij} := \Big\langle \frac{\mathbf{g}_i}{\|\mathbf{g}_i\|} \,, X \frac{\mathbf{g}_j}{\|\mathbf{g}_j\|} \Big\rangle$$

for $i, j \in [\![N-1]\!]$.

Remark A.1. It is easy to check that the matrix $M_{[\![m-1]\!]}$ is tridiagonal, i.e. $M_{ij} = 0$ if $|i-j| > 1$ and $i, j \in [\![m-1]\!]$. Hence, we call M the *tridiagonal matrix* associated with X around x.

Lemma A.2. *If M is the tridiagonal matrix of X then*

$$M_{i\,i+1} = \frac{\|\mathbf{g}_{i+1}\|}{\|\mathbf{g}_i\|}$$

for $i \in [\![m-2]\!]$.

Proof. We have

$$M_{i\,i+1} = \frac{\langle X\mathbf{g}_i\,,\mathbf{g}_{i+1}\rangle}{\|\mathbf{g}_i\|\|\mathbf{g}_{i+1}\|} = \frac{\langle X\mathbf{g}_i\,,Q_iX\mathbf{g}_i\rangle}{\|\mathbf{g}_i\|\|\mathbf{g}_{i+1}\|} = \frac{\langle Q_iX\mathbf{g}_i\,,Q_iX\mathbf{g}_i\rangle}{\|\mathbf{g}_i\|\|\mathbf{g}_{i+1}\|} = \frac{\|\mathbf{g}_{i+1}\|}{\|\mathbf{g}_i\|}. \qquad \square$$

Lemma A.3. *If $X = A$ is the adjacency matrix of a tree then the associated tridiagonal matrix M has zero diagonal.*

Proof. This is immediate from $M_{ii} = \langle A\mathbf{g}_i\,,\mathbf{g}_i\rangle/\|\mathbf{g}_i\|^2$ and the fact that the support of $\mathbf{g}_i$ consists of vertices whose distance to the root x has the same parity as i. $\square$

B. Tridiagonal matrix associated with a regular tree

In this appendix, we compute the tridiagonal matrix representation of $A = \mathrm{Adj}(G)$ if, in the vicinity of some vertex, G has the idealized graph structure described in Section 3.2. The section complements the explanations in Section 3.2 and the results are not used in the rest of the paper.

Throughout this section, we assume that there are $x \in [N]$ and $r \in \mathbb{N}$ such that G has the following structure in $B_r(x)$.

(i) The induced subgraph $G|_{B_r(x)}$ on $B_r(x)$ is a tree with root x.

(ii) The root x has D_x children and the vertices in $B_r(x) \setminus \{x\}$ have d children.

Lemma B.1 (Basis and tridiagonal representation)**.** *Let $\mathbf{s}_0, \dots, \mathbf{s}_r$ be the Gram-Schmidt orthonormalisation of $\mathbf{1}_x, A\mathbf{1}_x, \dots, A^r\mathbf{1}_x$. Then the following hold true*

(i) For all $i = 0, \dots, r$, we have

$$\mathbf{s}_i = |S_i(x)|^{-1/2}\mathbf{1}_{S_i(x)}.$$

(ii) Let $\mathbf{s}_{r+1}, \dots, \mathbf{s}_{N-1}$ be a completion of $\mathbf{s}_0, \dots, \mathbf{s}_r$ to an orthonormal basis of $\mathbb{R}^N$ and

$$M := S^*AS, \qquad S := (\mathbf{s}_0, \dots, \mathbf{s}_{N-1}),$$

the representation of A in this basis. Then the upper-left $(r+1)\times(r+1)$ block $M_{[\![r]\!]}$ of M is independent of $\mathbf{s}_{r+1}, \dots, \mathbf{s}_{N-1}$ and has the tridiagonal form

$$M_{[\![r]\!]} = \begin{pmatrix} 0 & \sqrt{D_x} & & & & \\ \sqrt{D_x} & 0 & \sqrt{d} & & & \\ & \sqrt{d} & 0 & \sqrt{d} & & \\ & & \sqrt{d} & 0 & \ddots & \\ & & & \ddots & \ddots & \sqrt{d} \\ & & & & \sqrt{d} & 0 \end{pmatrix}. \tag{B.1}$$

Note that the spectra of A and M coincide. We stress that, for all our arguments in the rest of the paper motivated the construction of M above only $M_{[\![r]\!]}$ plays a role. Therefore, the special choice of the basis vectors $\mathbf{s}_{r+1}, \dots, \mathbf{s}_{N-1}$ has no influence on these arguments.

Proof of Lemma B.1. For the proof of (i), we show inductively that

$$\mathbf{1}_{S_i(x)} = Q_i(A^i\mathbf{1}_x) \tag{B.2}$$

for $i = 0, 1, \ldots, r$, where Q_i is the orthogonal projection onto the orthogonal complement of $\mathbf{1}_x, \ldots, A^{i-1}\mathbf{1}_x$ for $i \geqslant 1$ and $Q_0 = I_N$. The initial step is trivial as $S_0(x) = \{x\}$.

For all $i \geqslant 1$, we have

$$Q_i(A^i \mathbf{1}_x) = Q_i(A\mathbf{1}_{S_{i-1}(x)})$$

as well as

$$A\mathbf{1}_{S_l(x)} = \begin{cases} \mathbf{1}_{S_1(x)}, & \text{if } l = 0, \\ \mathbf{1}_{S_2(x)} + D_x \mathbf{1}_x, & \text{if } l = 1, \\ \mathbf{1}_{S_{l+1}(x)} + d\mathbf{1}_{S_{l-1}(x)}, & \text{if } l \in [r-1] \setminus \{1\}. \end{cases} \tag{B.3}$$

Therefore, (B.2) follows immediately as $\mathbf{1}_{S_i(x)}$ and $\mathbf{1}_{S_j(x)}$ are orthogonal for $i \neq j$ and $\|\mathbf{1}_{S_i(x)}\| = |S_i(x)|^{1/2}$.

We start the proof of (ii) by concluding

$$\langle \mathbf{s}_i \,, A\mathbf{s}_j \rangle = |S_i(x)|^{-1/2} |S_j(x)|^{-1/2} \langle \mathbf{1}_{S_i(x)} \,, A\mathbf{1}_{S_j(x)} \rangle \tag{B.4}$$

for $i, j = 0, \ldots, r$ from (i). If $|i - j| \neq 1$ then this immediately yields $\langle \mathbf{s}_i \,, A\mathbf{s}_j \rangle = 0$. Moreover $|S_0(x)| = 1$ and $|S_i(x)| = D_x d^{i-1}$ for $i \geqslant 1$ to (B.4). For all $i, j = 0, 1, \ldots, r$, we have

$$\langle \mathbf{s}_i \,, A\mathbf{s}_j \rangle = \begin{cases} \sqrt{D_x}, & \text{if } |i-j| = 1 \text{ and } (i = 0 \text{ or } j = 0), \\ \sqrt{d}, & \text{if } |i-j| = 1 \text{ and } (i > 0 \text{ and } j > 0), \\ 0, & \text{if } |i-j| \neq 1. \end{cases}$$

This yields (ii) and, thus, completes the proof of Lemma B.1. □

C. Spectral properties of tridiagonal matrices

In this section, we analyse the spectral properties of tridiagonal matrices of the form

$$M(\alpha) = \begin{pmatrix} 0 & \sqrt{\alpha} & & & & \\ \sqrt{\alpha} & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & 1 & 0 & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & 0 \end{pmatrix} \in \mathbb{R}^{(r+1)\times(r+1)}, \tag{C.1}$$

where $\alpha > 0$ and $r \in \mathbb{N}$.

In the following lemma, we collect and prove a few spectral properties of $M(\alpha)$ for large r, in particular about its extreme eigenvalues and corresponding approximate eigenvectors. These results serve as motivation for the approximate eigenvectors introduced in Section 4 for large eigenvalues of the Erdős-Rényi graph. Moreover, along the following analysis preparing the proof of Lemma C.1 we introduce some natural concepts used throughout the arguments in this section.

Lemma C.1 (Eigenvalues and approximate eigenvectors of $M(\alpha)$)**.** *If $\alpha > 2$ then the following holds true:*

(i) (Extreme eigenvalues) The largest, $\lambda_1(M(\alpha))$, and the smallest eigenvalue, $\lambda_{r+1}(M(\alpha))$, of $M(\alpha)$ converge to $\Lambda(\alpha)$ and $-\Lambda(\alpha)$, respectively, as $r \to \infty$.

(ii) (Bulk eigenvalues) The eigenvalues $\lambda_2(M(\alpha)), \dots, \lambda_r(M(\alpha))$ *lie in* $[-2, 2]$.

(iii) (Approximate eigenvectors) Let $\mathbf{u} = (u_i)_{i=0}^r$ *and* $\mathbf{u}_- = ((-1)^i u_i)_{i=0}^r$ *have components*

$$u_0 \in \mathbb{R} \setminus \{0\}, \qquad u_1 := \left(\frac{\alpha}{\alpha - 1}\right)^{1/2} u_0, \qquad u_i := \left(\frac{1}{\alpha - 1}\right)^{(i-1)/2} u_1 \quad (i = 2, 3, \dots, r). \tag{C.2}$$

Then $\mathbf{u}$ *and* $\mathbf{u}_-$ *are approximate eigenvectors of* $M(\alpha)$ *corresponding to its largest and smallest eigenvalue, respectively.*

We now show how the eigenvectors of $M(\alpha)$ can be analysed by a transfer matrix approach. Let η be an eigenvalue of $M(\alpha)$ and $\mathbf{u} = (u_i)_{i=0}^r$ a corresponding eigenvector. The components of the eigenvector relation $M(\alpha)\mathbf{u} = \eta\mathbf{u}$ read as

$$\sqrt{\alpha} u_1 = \eta u_0, \qquad \sqrt{\alpha} u_0 + u_2 = \eta u_1, \qquad u_{i-1} + u_{i+1} = \eta u_i, \qquad u_{r-1} = \eta u_r \tag{C.3}$$

for $i = 2, \dots, r-1$. Hence, for $i = 1, \dots, r-1$, these relations are equivalent to

$$\begin{pmatrix} u_{i+1} \\ u_i \end{pmatrix} = T(\eta)^{i-1} \begin{pmatrix} u_2 \\ u_1 \end{pmatrix}, \tag{C.4}$$

where we introduced the 2×2 transfer matrix $T(\eta)$ defined through

$$T(\eta) = \begin{pmatrix} \eta & -1 \\ 1 & 0 \end{pmatrix}. \tag{C.5}$$

In the following, we will use these considerations if $|\eta| > 2$. In this case, we compute the spectrum and the eigenspaces of $T(\eta)$. The eigenvalues of $T(\eta)$ are $\gamma(\eta)$ and $\gamma(\eta)^{-1}$, where we defined

$$\gamma(\eta) := \frac{1}{2}\Big(\eta - \operatorname{sign}(\eta)\sqrt{\eta^2 - 4}\Big). \tag{C.6}$$

Moreover, the eigenspaces of $T(\eta)$ associated to $\gamma(\eta)$ and $\gamma(\eta)^{-1}$ are given by

$$\ker(T(\eta) - \gamma(\eta) I_2) = \operatorname{span}\left\{ \begin{pmatrix} \gamma(\eta) \\ 1 \end{pmatrix} \right\}, \quad \ker(T(\eta) - \gamma(\eta)^{-1} I_2) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ \gamma(\eta) \end{pmatrix} \right\}. \tag{C.7}$$

In the following, we denote the standard basis vectors of $\mathbb{R}^{r+1}$ by $\mathbf{e}_0, \dots, \mathbf{e}_r$.

Proof. We first prove (ii). To that end, we consider $M(\alpha)$ as a rank two perturbation of $M(1)$. It is well-known that

$$\operatorname{spec}(M(1)) = \left\{ 2\cos\left(\frac{\pi k}{r+2}\right) : k = 1, \dots, r+1 \right\} \subset [-2, 2].$$

This implies (ii) by Weyl's inequalities for eigenvalue perturbation as $\lambda_i(M(\alpha) - M(1)) = 0$ for any $i = 2, \dots, r$.

We now show (i) and (iii) simultaneously. Let $\mathbf{u}$ and $\mathbf{u}_-$ be defined as in (iii). We only focus on the largest eigenvalue of $M(\alpha)$ and $\mathbf{u}$. The same arguments work for the smallest eigenvalue and $\mathbf{u}_-$.

We set $\eta = \Lambda(\alpha)$ and obtain $\gamma(\eta) = (\alpha - 1)^{-1/2}$. Thus, $(u_{i+1}, u_i)^* \in \ker(T(\eta) - \gamma(\eta) I_2)$ for all $i = 1, \dots, r-1$. Hence, the equivalence between (C.3) for $i = 2, \dots, r-1$ and (C.4) implies that

$$(M(\alpha) - \eta I_{r+1})\mathbf{u} = (u_{r-1} - \eta u_r)\mathbf{e}_r.$$

Here, we also used $\eta = \Lambda(\alpha) = \alpha/\sqrt{\alpha - 1}$ and the relation between u_1 and u_0. Therefore, $\|(M(\alpha) - \Lambda(\alpha) I_{r+1})\mathbf{u}\| \to 0$ as $r \to \infty$. This completes the proof of Lemma C.1. □

For the formulation of the following proposition, we need some notation which we define now. For $\eta > 2$, we recall the definition of $\gamma(\eta)$ from (C.6) and introduce

$$\delta(m_{00}, m_{01}, m_{11}, m_{12}, \eta) = \frac{\gamma(\eta)(\eta - m_{00})(\eta - m_{11}) - \gamma(\eta) m_{01}^2 - m_{12}(\eta - m_{00})}{m_{01}^2 + \gamma(\eta)(\eta - m_{00}) m_{12} - (\eta - m_{00})(\eta - m_{11})} \tag{C.8}$$

whenever the denominator on the right-hand side is different from zero. For $\eta > 2$ and $\varepsilon > 0$, we also define

$$\gamma_{\geqslant}(m_{00}, m_{01}, m_{11}, m_{12}, \eta, \varepsilon) := \gamma(\eta)^{-1} - \frac{8(1+\eta)\varepsilon}{1-\gamma(\eta)^2}\Big(1 + 1 \vee (\delta(m_{00}, m_{01}, m_{11}, m_{12}, \eta))^2\Big)^{1/2} \tag{C.9}$$

Proposition C.2 (Bound on eigenvector of perturbations of $M(\alpha)$)**.** *Let $\widetilde{M}$ be a symmetric tridiagonal $(r+1)\times(r+1)$ matrix and $\mathbf{b} = (b_i)_{i \in [\![r]\!]} \in \mathbb{R}^{r+1}$. Let $\eta > 2$ and $\alpha > 2$. We set $\varepsilon := \|\widetilde{M} - M(\alpha)\|$, $\delta := \delta(\widetilde{M}_{00}, \widetilde{M}_{01}, \widetilde{M}_{11}, \widetilde{M}_{12}, \eta)$ and $\gamma_{\geqslant} = \gamma_{\geqslant}(\widetilde{M}_{00}, \widetilde{M}_{01}, \widetilde{M}_{11}, \widetilde{M}_{12}, \eta, \varepsilon)$. If the condition*

$$\eta^2 \geqslant 4 + \frac{4^5(1+\eta)^2\varepsilon^2}{(1-\gamma^2)^2}(1 + 1 \vee \delta^2) \tag{C.10}$$

is satisfied and $(\widetilde{M} - \eta I_{r+1})\mathbf{b} \in \operatorname{Span}\{\mathbf{e}_r\}$ then

$$\frac{(b_0)^2}{\|\mathbf{b}\|^2} \leqslant \frac{8(\widetilde{M}_{01}\widetilde{M}_{12})^2}{((\widetilde{M}_{01})^2 - (\eta - \widetilde{M}_{11})(\eta - \widetilde{M}_{00}) + \gamma \widetilde{M}_{12}(\eta - \widetilde{M}_{00}))^2}\Big[(\gamma_{\geqslant})^{-2r} \wedge \frac{1}{r-1}\Big].$$

and

$$\gamma_{\geqslant} \geqslant 1 + \frac{1}{2}(\gamma^{-1} - 1) \geqslant 1. \tag{C.11}$$

Proof. As $(\widetilde{M} - \eta I_{r+1})\mathbf{b} \in \operatorname{Span}\{\mathbf{e}_r\}$ we have

$$\widetilde{M}_{00} b_0 + \widetilde{M}_{01} b_1 = \eta b_1, \quad \widetilde{M}_{i\,i-1} b_{i-1} + \widetilde{M}_{ii} b_i + \widetilde{M}_{i\,i+1} b_{i+1} = \eta b_i \tag{C.12}$$

for any $i = 1, \dots, r-1$. Hence,

$$\begin{pmatrix} b_{i+1} \\ b_i \end{pmatrix} = T_i \begin{pmatrix} b_i \\ b_{i-1} \end{pmatrix}, \qquad T_i := \frac{1}{\widetilde{M}_{i\,i+1}} \begin{pmatrix} \eta - \widetilde{M}_{i\,i} & -\widetilde{M}_{i\,i-1} \\ \widetilde{M}_{i\,i+1} & 0 \end{pmatrix} \tag{C.13}$$

for $i = 1, \dots, r-1$. For $i \geqslant 2$, we define $R_i := T_i - T$, where $T = T(\eta)$ is defined as in (C.5). As $\varepsilon = \|\widetilde{M} - M(\alpha)\|$, we have $\|R_i\| \leqslant 2(1+\eta)\varepsilon$ uniformly for $i \geqslant 2$.

For each i, we denote by p_i and q_i the first and second component of $(b_{i+1}, b_i)^*$ in the eigenbasis of T, respectively. That is

$$\begin{pmatrix} p_i \\ q_i \end{pmatrix} := V^{-1} \begin{pmatrix} b_{i+1} \\ b_i \end{pmatrix}, \qquad V = \begin{pmatrix} \gamma & 1 \\ 1 & \gamma \end{pmatrix}, \tag{C.14}$$

where $\gamma = \gamma(\eta)$ with the definition of $\gamma(\eta)$ from (C.6). The fact that $V^{-1}TV$ is diagonal can be easily read off from (C.7).

We will now show that

$$\frac{|p_i|}{|q_i|} \leqslant 1 \vee \Big(\frac{|p_1|}{|q_1|}\Big), \qquad |q_i| \geqslant (\gamma_{\geqslant})^{i-1}|q_1| \tag{C.15}$$

for all i by induction on i. The assertion is trivial for $i = 1$. From (C.14) and (C.13), we conclude

$$\begin{pmatrix} p_{i+1} \\ q_{i+1} \end{pmatrix} = V^{-1}\Big(V\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix}V^{-1} + R_{i+1}VV^{-1}\Big)\begin{pmatrix} b_{i+1} \\ b_i \end{pmatrix} = \Big(\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix} + V^{-1}R_{i+1}V\Big)\begin{pmatrix} p_i \\ q_i \end{pmatrix}.$$

Estimating the first component of this relation implies

$$
\begin{aligned}
|p_{i+1}| &\leqslant \gamma|p_i| + \left\| V^{-1}R_{i+1}V \begin{pmatrix} p_i \\ q_i \end{pmatrix} \right\| \\
&\leqslant \left(\gamma \frac{|p_i|}{|q_i|} + \|V^{-1}R_{i+1}V\| \left(1 + \left(\frac{p_i}{q_i} \right)^2 \right)^{1/2} \right) |q_i| \\
&\leqslant \left(1 \vee \frac{|p_1|}{|q_1|} \right) \left(\gamma + \sqrt{2}\|V^{-1}R_{i+1}V\| \right) |q_i|,
\end{aligned}
$$

where we used that $1 + p_i^2/q_i^2 \leqslant 2(1 \vee p_1^2/q_1^2)$ by the induction hypothesis in the last step. Similarly, we bound the second component from below and obtain

$$
\begin{aligned}
|q_{i+1}| &\geqslant \left(\gamma^{-1} - \|V^{-1}R_{i+1}V\| \left(\left(\frac{p_i}{q_i} \right)^2 + 1 \right)^{1/2} \right) |q_i| \\
&\geqslant \left(\gamma^{-1} - \|V^{-1}R_{i+1}V\| \left(1 + 1 \vee \left(\frac{p_1}{q_1} \right)^2 \right)^{1/2} \right) |q_i|
\end{aligned}
\tag{C.16}
$$

due to the induction hypothesis.

By dividing the upper bound on $|p_{i+1}|$ by the lower bound on $|q_{i+1}|$, we see that the induction step for the first bound in (C.15) is shown if

$$
\gamma + \sqrt{2}\|V^{-1}R_{i+1}V\| \leqslant \gamma^{-1} - \|V^{-1}R_{i+1}V\| \left(1 + 1 \vee \left(\frac{p_1}{q_1} \right)^2 \right)^{1/2}. \tag{C.17}
$$

This bound will be deduced from the first condition in (C.10) next. To that end, we first compute p_1/q_1. The definition of p_1 and q_1 in (C.14) yields $p_1 = (-b_1 + \gamma b_2)/(\gamma^2 - 1)$ and $q_1 = (-b_2 + \gamma b_1)/(\gamma^2 - 1)$. We use the first relation in (C.12) and the second relation in (C.12) with $i = 1$ to express b_1 and b_2 in terms of b_0. Then an easy computation shows that

$$
p_1 = \frac{\gamma b_2 - b_1}{\gamma^2 - 1} = \frac{b_0 \big(\gamma(\eta - \widetilde{M}_{11})(\eta - \widetilde{M}_{00}) - \gamma(\widetilde{M}_{01})^2 - \widetilde{M}_{12}(\eta - \widetilde{M}_{00}) \big)}{(\gamma^2 - 1)\widetilde{M}_{01}\widetilde{M}_{12}}, \tag{C.18a}
$$

$$
q_1 = \frac{-b_2 + \gamma b_1}{\gamma^2 - 1} = \frac{b_0 \big((\widetilde{M}_{01})^2 - (\eta - \widetilde{M}_{11})(\eta - \widetilde{M}_{00}) + \gamma \widetilde{M}_{12}(\eta - \widetilde{M}_{00}) \big)}{(\gamma^2 - 1)\widetilde{M}_{01}\widetilde{M}_{12}}. \tag{C.18b}
$$

Therefore, we obtain

$$
\frac{p_1}{q_1} = \frac{-b_1 + \gamma b_2}{-b_2 + \gamma b_1} = \delta := \delta(\widetilde{M}_{00}, \widetilde{M}_{01}, \widetilde{M}_{11}, \widetilde{M}_{12}, \eta),
$$

where we used the function δ defined in (C.8). Thus, as $\|V^{-1}R_{i+1}V\| \leqslant 4\|R_{i+1}\|/(1 - \gamma^2) \leqslant 8(1 + \eta)\varepsilon/(1 - \gamma^2)$ the definition of $\gamma(\eta)$ in (C.6) shows that (C.17) is a consequence of (C.10). This completes the induction step for the first estimate in (C.15).

From (C.16), $p_1/q_1 = \delta$ and $\|V^{-1}R_{i+1}V\| \leqslant 8(1 + \eta)\varepsilon/(1 - \gamma^2)$, we deduce $|q_{i+1}| \geqslant \gamma_{\geqslant}|q_i|$. Thus, we have completed the proof of (C.15).

We now prove that (C.10) also implies the lower bound on $\gamma_{\geqslant}$ in (C.11). The definition of $\gamma(\eta)$ in (C.6) yields $2(\gamma^{-1} - 1) \geqslant (\gamma^{-1} - \gamma) = \sqrt{\eta^2 - 4}$. Thus, we obtain from (C.10) that

$$
\begin{aligned}
\gamma_{\geqslant} - 1 - \frac{1}{2}(\gamma^{-1} - 1) &= \frac{1}{2}\left[\gamma^{-1} - 1 - \frac{16(1 + \eta)\varepsilon}{1 - \gamma^2} \left(1 + 1 \vee \left(\frac{p_1}{q_1} \right)^2 \right)^{1/2} \right] \\
&\geqslant \frac{1}{4}\left[\sqrt{\eta^2 - 4} - \frac{32(1 + \eta)\varepsilon}{1 - \gamma^2} \left(1 + 1 \vee \left(\frac{p_1}{q_1} \right)^2 \right)^{1/2} \right]
\end{aligned}
$$

Owing to (C.10), the right-hand side is positive. This immediately implies (C.11).

Owing to the second bound in (C.15), we have

$$\begin{aligned} 2\sum_{i=0}^{r}(b_i)^2 &\geqslant \sum_{i=0}^{r-1}\left\|\begin{pmatrix} b_{i+1} \\ b_i \end{pmatrix}\right\|^2 \\ &= \sum_{i=0}^{r-1}\left\|VV^{-1}\begin{pmatrix} b_{i+1} \\ b_i \end{pmatrix}\right\|^2 \\ &\geqslant \frac{1}{\|V^{-1}\|^2}\sum_{i=1}^{r-1}\left\|\begin{pmatrix} p_i \\ q_i \end{pmatrix}\right\|^2 \\ &\geqslant \frac{(\gamma^2-1)^2}{4}|q_1|^2\Big[(\gamma_{\geqslant})^{2r} \vee (r-1)\Big]. \end{aligned} \tag{C.19}$$

Here, we pulled V out of the norm in the third step and used (C.14). The fourth step is a consequence of $\|V^{-1}\| \leqslant 2(1-\gamma^2)^{-1}$, estimating the norm by its second component and using the second bound in (C.15) as well as $\gamma_{\geqslant} \geqslant 1$ due to (C.11).

Finally, applying (C.18b) to (C.19) completes the proof of Proposition C.2. □

D. Degree distribution of the Erdős-Rényi graph

The content of this section is standard, and we include it for completeness and the reader's convenience. It is essentially contained in [4, Chapter 3]. We do not aim for sharp estimates of the error probabilities; instead, our goal here is to collect basic qualitative facts about the behaviour of the largest degrees of an Erdős-Rényi graph, which, using Theorem 2.1, can be used to understand the key properties of the extremal eigenvalues. We recall the normalized degree (2.1) and the random permutation (2.3).

To formulate qualitative statements conveniently, we use the symbol $o(1)$ to denote any function of N that converges to zero, and say that an N-dependent event $\Xi \equiv \Xi_N$ holds with high probability if $\mathbb{P}(\Xi) = 1 - o(1)$.

The distribution of the largest degrees is best analysed using the function

$$f_d(\alpha) := d(\alpha \log \alpha - \alpha + 1) + \frac{1}{2}\log(2\pi\alpha d) \tag{D.1}$$

for $\alpha \geqslant 1$. For its interpretation, we note that if $Y \overset{\mathrm{d}}{=} \text{Poisson}(d)$ then by Stirling's formula we have for any $k \in \mathbb{N}$

$$\mathbb{P}(Y = k) = \exp\bigg(-f_d(k/d) + O\Big(\frac{1}{k}\Big)\bigg).$$

It is easy to see that the function $f_d : [1, \infty) \to \big[\frac{1}{2}\log(2\pi d), \infty\big)$ is bijective and increasing. Therefore there is a universal constant $C > 0$ such that for $1 \leqslant l \leqslant \frac{N}{C\sqrt{d}}$ such that the equation

$$f_d(\beta) = \log(N/l)$$

has a uniquel solution $\beta \equiv \beta_l(d)$. The interpretation of β is the typical location of $\alpha_{\sigma(l)}$. By the implicit function theorem, we find that β_l on the interval $\big(0, \frac{N^2}{Cl^2}\big]$ is a decreasing bijective function.

We are interested in normalized degrees greater than or equal to 2. This motivates the definition

$$\mathcal{L}(d) := \max\{l \geqslant 1 : \beta_l(d) \geqslant 2\},$$

whose interpretation is the typical number of normalized degrees greater than or equal to 2. By definition, $\beta_l(d) \geqslant 2$ for all $l \leqslant \mathcal{L}(d)$. Note that $\mathcal{L}(d)$ is nonzero if and only if $d \leqslant d_*$, where d_* is defined as the unique solution of $\beta_1(d_*) = 2$. More explicitly, d_* satisfies $f_{d_*}(2) = \log N$.

Proposition D.1. *Let $\xi \equiv \xi_N$ be a positive sequence tending to ∞. If $1 \leqslant d \leqslant d_*$ and $1 \leqslant l \leqslant \mathcal{L}(d)$ then with high probability we have*

$$|\alpha_{\sigma(l)} - \beta_l(d)| \leqslant \frac{1 \vee (\xi / \log \beta_l(d))}{d} \,. \tag{D.2}$$

If $d > d_*$ then with high probability we have

$$\alpha_{\sigma(1)} \leqslant 2 + \frac{\xi}{d} \,. \tag{D.3}$$

Proof. Throughout the proof we suppose that $2 \leqslant \alpha \leqslant \frac{\sqrt{N}}{Cd}$ for some large enough universal constant C. From the definition of $\beta_l(d)$, it is easy to check that this condition is satisfied for $\alpha = \beta_l(d)$ for $1 \leqslant d \leqslant d_*$ and $1 \leqslant l \leqslant \mathcal{L}(d)$. The proof of (D.2) consists of an upper and a lower bound. The former is proved using a first moment method and the latter using a second moment method. We make use of the counting function $\mathcal{N}_t := \sum_{x \in [N]} \mathbb{1}_{D_x \geqslant t}$. Note that by Poisson approximation of the binomial random variable $D_x = d\alpha_x$, see [2, Lemma 3.3], there is a universal constant C such that

$$C^{-1} N \mathrm{e}^{-f_d(\alpha)} \leqslant \mathbb{E}\mathcal{N}_{\alpha d} \leqslant C N \mathrm{e}^{-f_d(\alpha)} \,. \tag{D.4}$$

Let $1 \leqslant d \leqslant d_*$ and $1 \leqslant l \leqslant \mathcal{L}(d)$. We begin by proving an upper bound on $\alpha_{\sigma(1)} = D_{\sigma(1)}/d$. Using (D.4) we get

$$\mathbb{P}(\alpha_{\sigma(l)} \geqslant \alpha) = \mathbb{P}(\mathcal{N}_{\alpha d} \geqslant l) \leqslant \frac{\mathbb{E}\mathcal{N}_{\alpha d}}{l} \leqslant \frac{CN}{l} \mathrm{e}^{-f_d(\alpha)} \,. \tag{D.5}$$

Next, we prove a lower bound on $\alpha_{\sigma(l)}$. Suppose that $\mathbb{E}\mathcal{N}_{\alpha d} \geqslant 2l$. Then using a second moment method, we find

$$\mathbb{P}(\alpha_{\sigma(l)} \geqslant \alpha) = \mathbb{P}(\mathcal{N}_{\alpha d} \geqslant l) \geqslant \mathbb{P}(|\mathcal{N}_{\alpha d} - \mathbb{E}\mathcal{N}_{\alpha d}| < \mathbb{E}\mathcal{N}_{\alpha d}/2) \geqslant 1 - \frac{4 \operatorname{Var}(\mathcal{N}_{\alpha d})}{(\mathbb{E}\mathcal{N}_{\alpha d})^2} \,.$$

By [4, Lemma 3.11] we have $\operatorname{Var}(\mathcal{N}_{\alpha d}) \leqslant C \mathbb{E}\mathcal{N}_{\alpha d}$ for some universal constant C, which yields

$$\mathbb{P}(\alpha_{\sigma(l)} \geqslant \alpha) \geqslant 1 - \frac{C}{\mathbb{E}\mathcal{N}_{\alpha d}} \geqslant 1 - \frac{C}{N \mathrm{e}^{-f_d(\alpha)}} \,, \tag{D.6}$$

where we used (D.4). From (D.5) we conclude that $\alpha_{\sigma(l)} \leqslant \alpha$ with high probability if

$$f_d(\alpha) - \log(N/l) \to \infty \quad \text{as} \quad N \to \infty \,. \tag{D.7}$$

From (D.6) we conclude that $\alpha_{\sigma(l)} \geqslant \alpha$ with high probability if $\mathbb{E}\mathcal{N}_{\alpha d} \geqslant 2l$ and $f_d(\alpha) - \log N \to -\infty$ as $N \to \infty$. By (D.4), both of these conditions are satisfied if

$$f_d(\alpha) - \log(N/l) \to -\infty \quad \text{as} \quad N \to \infty \,. \tag{D.8}$$

Now (D.2) follows easily by choosing $\alpha = \beta_l(d) \pm \frac{1 \vee (\xi / \log \beta_l(d))}{d}$, using that $f_d'(\alpha) \geqslant d \log \alpha$.

The proof of (D.3) is analogous to the the proof of the upper bound in (D.2), and we omit the details. □

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