Analysis of the one dimensional inhomogeneous Jellium model with the Birkhoff-Hopf Theorem

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Abstract

We use the Hilbert distance on cones and the Birkhoff-Hopf Theorem to prove decay of correlation, analyticity of the free energy and a central limit theorem in the one dimensional Jellium model with non constant density charge background, both in the classical and quantum cases.

Contents

1 Introduction

The Jellium model describes a system of electrons interacting with each other in a continuous background of opposite charge. It is a very fundamental system in quantum chemistry and condensed matter physics [15, 30, 8]. The model has been initially introduced by Wigner [33]. In (quasi-)one dimension it has then been rigorously studied when the background is uniform by Kunz [26], Brascamp-Lieb [9], Aizenman-Martin [2] and many others [6, 4, 10, 17, 24, 1, 23, 28]. This model is known to reveal a symmetry breaking called the Wigner crystal. One major difficulty is that the Coulomb potential is long range. In dimension one, the interaction is like $-|x-y|$ and therefore the force between two particles does not depend on their mutual distance which simplifies a lot the problem.

In this paper we study the inhomogeneous Jellium model in which the background is not constant. The inhomogeneous case is very important for applications, at least in three dimensions [16, 21, 29]. The Wigner crystal still appears for a periodic background, provided that the charge in one period is equal to the charge of the particles. Here we consider any background in one dimension and the system will not necessarily be crystallized.

One of the most important properties of the constant background model is that the classical partition function for N particles can be written in the form

$$\mathcal{Z}_N(\beta) = \langle a, T(\beta)^N b\rangle$$

where $T(\beta)$ is a compact operator with positive kernel in some L^2 space, which depends smoothly on the inverse temperature β. By the Krein-Rutmann Theorem [3], $T(\beta)$ has a unique largest eigenvalue $\lambda(\beta) > 0$ which is always non degenerate, hence is also a smooth function of β. As a consequence, the free energy per particle behaves as

$$f_N(\beta) = -\frac{1}{N\beta}\log(\mathcal{Z}_N(\beta)) = -\frac{1}{\beta}\log(\lambda(\beta)) - \frac{1}{\beta N}\log(\langle a, v\rangle\langle v, b\rangle) + O(\kappa^N)$$

for large N, where v is the unique positive eigenvector associated with $\lambda(\beta)$ and $\kappa < 1$. In fact, $T(\beta)^N/\lambda(\beta)^N$ is close to the rank-one projection on v and this can also be used to prove the decay of correlations.

In the inhomogeneous case, the classical partition function takes the form

$$\mathcal{Z}_N(\beta) = \left\langle a, \prod_{0\leq i\leq N-1} T_i(\beta) b \right\rangle \tag{1}$$

where the transitive operators $T_i(\beta)$ are no longer equal to each other. Our goal is to generalize the results proved in the homogeneous case to the inhomogeneous case. For this we will replace the spectral approach based on the Krein Rutmann theorem by the Birkhoff-Hopf theorem [5, 22]. The main idea behind this method is to quantify how a product of many operators with positive kernels can be well approximated by a rank-one operator. A main tool is the so-called Hilbert distance on cones, a concept which will be discussed at length later on.

Using these tools we will prove the decay of correlations and the smoothness of the free energy in the inhomogeneous Jellium model. In the classical case, we can essentially handle any background, but in the quantum case we require it to be close to a constant. Our method is general and can be applied to other one dimensional inhomogeneous systems in statistical physics like the Ising model. It could also be useful for log gases [12, 14]. For this reason, we will present the theory in the abstract framework of cones on any Banach spaces, in a form which is well suited to the setting of statistical physics.

Our paper is organized as follows. We first describe in Section 2 the Jellium model and state our main results both for the classical and the quantum cases. We then introduce in Section 3.1 the framework required to state the Birkhoff-Hopf Theorem. Afterward, we suggest a new formulation of weak ergodicity using rank-one operators and prove it in Section 3.3. As it is shown in Section 3.4, the rank-one approximation implies that the k-particle marginals are well approximated by (independent) products of the 1-particle marginals. In Section 3.5 we prove the regularity of the abstract free energy. Finally we deal with the inhomogeneous Jellium model. Section 4.1 and Section 4.2 are dedicated to the proof of the classical and quantum cases, respectively. The main result of theses two sections is the construction of an appropriate cone such that the theorems of the previous sections can be applied.

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2 The Jellium model

In this section, we present the Jellium model and state all our results. The proofs will be given in Section 4.1 and Section 4.2.

2.1 The classical Jellium model

2.1.1 Mathematical formalism

We consider N particles of negative charges $q_1, \dots, q_N$ placed on a line $-L < x_1 < x_2 < \dots < x_N < L$ in an inhomogeneous fixed density of charge $\rho \in L^1([-L;L])$ such that $\int_{-L}^{L} \rho(s)ds = -\sum_{i=1}^{N} q_i$. The one dimensional solution of $u'' = 2\delta_0$ is $u(x) = -|x|$, which gives us the total energy of the system

$$E(x_1, x_2, \dots ,x_N) = -\frac{1}{2} \iint_{[-L,L]^2} \rho(y_1)\rho(y_2)|y_1 - y_2| dy_1 dy_2 \\ - \frac{1}{2} \sum_{1 \le i,j \le N} q_i q_j |x_i - x_j| + \sum_{i=1}^{N} q_i \int_{-L}^{L} \rho(y)|x_i - y| dy.$$

The first term is the background-background interaction, the second term accounts for the electron-electron interaction and the third term for the background-electron interaction.

Let us first calculate the state of minimum energy. For each particle i the position $\tilde{x}_i$ which minimizes the energy is such that

$$\int_{-L}^{\tilde{x}_i} \rho(y)dy = \sum_{1 \le j < i} q_j + \frac{q_i}{2}.$$

It is the condition that for each particle there is the same amount of charge on its right side and on its left side, such that the particle is at equilibrium. In the homogeneous case, we have for any i, $\rho|\tilde{x}_{i+1} - \tilde{x}_i| = q$. Therefore at $T = 0$, the electrons are located on $\frac{q}{\rho}\mathbb{Z}$ (the Wigner cristal). But for a general background the lattice is not necessarily a solution.

We subtract the minimum of the energy and rewrite it as

$$E(x_1, \dots, x_N) = E(\tilde{x}_1, \dots, \tilde{x}_N) + 2 \sum_{i=1}^{N} q_i \int_{\tilde{x}_i}^{x_i} \rho(y)(y - x_i) dy.$$

We denote by

$$U_i(s) = -2q_i \int_{\tilde{x}_i}^{\tilde{x}_i + s} (y - \tilde{x}_i - s)\rho(y) dy$$

the potential felt by the i^{th}−particle around its stable position.

We are interested in the canonical model at positive temperature. The position of the particles x_i are now random and the probability of a set of positions $(x_i)_{i=1,\dots,N}$ is proportional to $e^{-\beta E(x_1,\cdots,x_N)}$ (Gibbs measure).

The relevant physical properties of the system are obtained from the partition function given by

$$\mathcal{Z}_N(\beta) = e^{-\beta E(\tilde{x}_1,\dots,\tilde{x}_N)} \int \cdots \int_{-L<x_1<x_2<\dots<x_N<L} \prod_{i=1}^{N} e^{-2\beta q_i \int_{\tilde{x}_i}^{x_i} \rho(y)(y-x_i)dy} dx_i$$

and its free energy per particle

$$f_N(\beta) = -\frac{1}{N\beta}\log(\mathcal{Z}_N(\beta)).$$

We also introduce the marginals $\rho_I(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$, for the probability of the positions of the k particles of the subset $I = \{i_1, i_2, \cdots, i_k\} \subset \{1, \cdots, N\}$. More rigorously, it is the unique function such that for all test functions $g \in L^\infty([-L, L]^k)$,

$$\begin{aligned}
&\iiint g(x_{i_1}, x_{i_2}, \cdots, x_{i_k})\rho_k(x_{i_1}, \cdots, x_{i_k})dx_{i_1} \cdots dx_{i_k} \\
&\quad = \frac{e^{-\beta E(\tilde{x}_1, \ldots, \tilde{x}_N)}}{\mathcal{Z}_N(\beta)} \iint_{-L<x_1<x_2<\ldots<x_N<L} g(x_{i_1}, \cdots, x_{i_k}) \times \\
&\qquad \times \prod_{k=1}^{N} e^{-2\beta q_i \int_{\tilde{x}_i}^{x_i} \rho(y)(y-x_i)dy} dx_i.
\end{aligned}$$

Our main interest is to know whether the particles are strongly correlated or not. This can be quantified by looking at the truncated correlation functions, which we introduce below.

Definition 1. (Cluster property) We say that

- the particles are *independent* if

$$\rho_{\{1,\cdots,N\}}(x_1, ..., x_N) = \prod_{i=1}^{n} \rho(x_i),$$

- the particles satisfy a *cluster property* if there exists $I \cup J = \{1, \cdots, n\}, I \cap J = \emptyset$ such that

$$\rho_{\{1,\cdots,N\}}(x_1, ..., x_n) = \rho_{|I|}((x_i)_{i\in I})\rho_{|J|}((x_j)_{j\in J}).$$

In order to characterize the "clusters" we introduce the truncated marginal:

Definition 2. (Truncated marginal) The *truncated marginals* ρ_k^T are defined recursively as follows:

$$\rho_J^T(x_{j_1}, ..., x_{j_k}) = \rho_J(x_{j_1}, ..., x_{j_k}) - \sum_{I_1\cup I_2\cup\ldots\cup I_r = J} \prod_{l=1}^{r} \rho_{I_l}^T((x_i)_{i\in I_l}).$$

The truncated marginals appear to be the good indicator for clustering properties. Indeed we have the following proposition.

Proposition 3. *If* $\rho_n(x_1, ..., x_n) = \rho_{|I|}((x_i)_{i\in I})\rho_{|J|}((x_j)_{j\in J})$ *then for all* I' *such that* $I' \cap I \neq \emptyset$ *and* $I' \cap J \neq \emptyset$ *then* $\rho_{|I'|}^T((x_i)_{i\in I'}) = 0$.

For the reader's convenience we have written the proof of Proposition 3 in Appendix A. We are now ready to state our main results.

2.1.2 Main results

In the classical case we make the following assumptions:

- (H1) There exist $q, Q > 0$ such that for all $i, 0 < q \leq q_i \leq Q$.
- (H2) There exists $0 < m < M$ such that for all $t \in [-L, L], m \leq \rho(t) \leq M$.

These assumptions (H1,H2) imply the following bounds for the potential:

$$U_i(s) \geq qms^2$$

and

$$\frac{d}{ds} U_i(s) \geq smq.$$

Our first result is to be understood as follows: If we consider particles which are far away from each other (meaning that there are a lot of other particles between them) then the marginal is exponentially close to the independent marginal. We also get the cluster property: if groups of particles are far from each others, then the marginal is exponentially close to the independent cluster marginal.

Theorem 4. *For any $\beta > 0$, there exists $\kappa < 1$ such that for any $I \subset \{0, \cdots, N\}$, $|I| = k$, we have*

$$\left| \rho_I(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) - \prod_{i_l \in I} \rho_{\{i_l\}}(x_{i_l}) \right| \leq C_k \kappa^d$$

for some $C_k > 0$, provided that between any two consecutive particle in I there are at least d others particles (in practice take $d = \inf |i_l - i_{l+1}| - 1$). Also we have

$$|\rho_I^T(x_{i_1}, x_{i_2}, \ldots, x_{i_k})| \leq C_k \kappa^D$$

when there exist two consecutive particles in I with at least D others particles between them (in practice take $D = \max |i_l - i_{l+1}| - 1$).

Our next result concerns the regularity of the free energy, which is a fundamental property for one-dimensional systems in statistical physics.

Theorem 5. *For any $\beta_0 > 0$, there exists $\Delta\beta > 0$ such that the free energy is smooth on $[\beta_0 - \Delta\beta, \beta_0 + \Delta\beta]$ uniformly on N. More precisely we have*

$$|\frac{d^k}{d\beta^k} f_N| \leq M_k,$$

with $M_k > 0$ independent of N and for all $\beta \in [\beta_0 - \Delta\beta, \beta_0 + \Delta\beta]$.

In the proof we will show the following estimate on M_k:

$$M_k \leq k! c^k \tag{2}$$

for some $c > 0$. From this bound we obtain the analyticity of the limiting free energy, when this limit exists.

Theorem 6. *For any $\beta_0 > 0$, there exists $\Delta\beta > 0$ such that if there exists f such that a $f_{N_k} \to f$ on $[\beta_0 - \Delta\beta, \beta_0 + \Delta\beta]$ for a subsequence $N_k \to \infty$, then f is real analytic on $[\beta_0 - \Delta\beta, \beta_0 + \Delta\beta]$.*

As a corollary, the system will not reveal any phase transition for $\beta \neq \infty$.

Corollary 7. *If the charge background is periodic or if it is constructed randomly with an ergodic process, there exists a limiting function f such that $f_N \to f$ (almost surely in the ergodic case) and f is real analytic on $(0, \infty)$.*

This generalizes the results of Kunz [26].

2.2 The quantum model

2.2.1 Mathematical formalism

We now give our results for the quantum problem. In the classical case, we neglect the kinetic energy because in phase space momentum and position are independent for the Gibbs measure. This is no longer true in the quantum case and we have to consider the whole N-particle Hamiltonian

$$H = -\frac{1}{2}\sum_{i=1}^{N} \partial_{x_i}^2 + E(x_1, \cdots, x_N).$$

For simplicity we choose Dirichlet boundary conditions at the two ends $\mp L$. The quantum fermionic canonical function is

$$\mathcal{Z}_N^Q(\beta) = \mathrm{Tr}(\exp(-\beta H))$$

and the free energy is

$$f_N^Q(\beta) = -\frac{1}{\beta N}\log(\mathcal{Z}_N^Q(\beta)).$$

We have the following Feynman-Kac formula [25] for the partition function $\mathcal{Z}_N^Q(\beta)$.

Proposition 8. *(Feynman Kac formula) We have*

$$\mathcal{Z}_N^Q(\beta) = \int_{\mathcal{X}} \mu_{x_1 x_1} \ldots \mu_{x_N x_N} (e^{-\int_0^\beta U(\gamma_1(t), \ldots, \gamma_N(t))dt} 1_{(\gamma_1, \cdots, \gamma_N) \in W_N}) dx_1 \ldots dx_N \tag{3}$$

and

$$\rho(\boldsymbol{x}; \boldsymbol{y}) = \frac{1}{Z_N} \mu_{x_1 y_1} \times \mu_{x_2, y_2} \times \ldots \times \mu_{x_N, y_N} (e^{-\int_0^\beta U(\gamma_1(t), \ldots, \gamma_N(t))dt})$$

where $\mathcal{X} = \{(x_1, \cdots, x_N) : -L < x_1 < \cdots < x_N < L\}$,

$$W_N = \{(\gamma_1, \ldots, \gamma_N) | \forall t \in [0, \beta] : -L < \gamma_1(t) < \gamma_2(t) < \ldots < \gamma_N(t) < L\}$$

is the Weyl chamber and $\mu_{x,y}$ are the probability measures of a Brownian bridge from x to y of length β.

The random system we study in the quantum model is no longer the positions $(x_i)_{i\leq N}$ but rather the paths $(\gamma_i)_{i\leq N}$. We define the extended marginals on the set of paths $\rho^\Gamma(\gamma_1, ..., \gamma_N)$ and we are able to apply the theorems of Section 3.4 in this set up. However, for simplicity we will only states the results on the position marginals $\rho_k(x_{i_1}, ..., x_{i_k})$ which satisfy, for any bounded function $g : [-L, L]^N \to \mathbb{R}$,

$$\int \cdots \int \rho_k(x_{i_1}, \cdots, x_{i_k}) g(x_{i_1}, \cdots, x_{i_k}) dx_1 \cdots dx_k$$
$$= \frac{1}{Z_N^Q(\beta)} \int_{-L<x_1<\cdots<x_N<L} \mu_{x_1x_1} \cdots \mu_{x_Nx_N} (e^{-\int_0^\beta U(\gamma_1(t), \ldots, \gamma_N(t))dt} 1_{(\gamma_1, \cdots, \gamma_N) \in W_N})$$
$$g(x_{i_1}, \cdots, x_{i_k}) dx_1 \ldots dx_N$$

2.2.2 Main results

Unfortunately, in the quantum case we are only able to prove a result in a perturbation regime where ρ and the q_i are almost constant. We therefore make the following assumptions :

- (HQ1) $q(1-\epsilon) \leq q_i \leq q(1+\epsilon)$ for all i.
- (HQ2) $\rho(1-\epsilon) \leq \rho(t) \leq \rho(1+\epsilon)$ for all t.

Theorem 9. *For any $\beta > 0$, under condition (HQ1-2) for $\epsilon > 0$ small enough, there exists $\kappa < 1$, such that for all $I \subset \{1, \cdots, N\}$, $|I| = k$,*

$$|\rho_I(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) - \prod_{i_l \in I} \rho_{\{i_l\}}(x_{i_l})| \leq C_k \kappa^d$$

for some $C_k > 0$, if between any two consecutive particle in I there are at least d others particles (in practice take $d = \inf |i_l - i_{l+1}| - 1$). On the other hand,

$$|\rho_I^T(x_{i_1}, x_{i_2}, \ldots, x_{i_k})| \leq C_k \kappa^D$$

if there exists two consecutive particles in I with at least D others particles between them (in practice take $D = \max |i_l - i_{l+1}| - 1$).

As in the classical case we obtain the regularity of the partition function f_N^Q.

Theorem 10. *For any $\beta_0 > 0$, there exists $\Delta\beta > 0$ such that under condition (HQ1-2) with $\epsilon > 0$ small enough, the free energy is C^∞ on $[\beta_0 - \Delta\beta, \beta_0 + \Delta\beta]$ and for all k we have*

$$|\frac{d^k}{d\beta^k} f_N^Q| \leq M_k$$

with M_k independent of N.

Finally we can prove analyticity of the free energy with the same estimate as (2).

Theorem 11. *For any $\beta_0 > 0$, there exists $\Delta\beta > 0$ such that under condition (HQ1-2), for $\epsilon > 0$ small enough, if f_{N_k} admits a limit f for a subsequence $N_k \to \infty$, then f is real analytic on $[\beta_0 - \Delta\beta, \beta_0 + \Delta\beta]$.*

Corollary 12. *We make the same assumptions as in Theorem 9. If the charge background is periodic or if it is constructed randomly with an ergodic process, there exists a limiting function f such that $f_N \to f$ (almost surely in the ergodic case) and f is real analytic.*

3 General theory to apply the Birkhoff-Hopf theorem

The Birkhoff-Hopf theorem has been used for instance to study non linear integrable equations, weak ergodic theorems, or the so-called DAD problem [7].

We first introduce the notion of cone and the Hilbert distance. In this set up we can state the Birkhoff-Hopf theorem. Then we prove Theorem 4 and Theorem 5 with the extra assumption of strictly contracting operators.

3.1 Framework and Birkhoff-Hopf theorem.

We follow [13] for the notation and we refer to this paper for a proof of the Birkhoff Hopf theorem (Theorem 18 bellow). Let E be a real linear Banach space.

Definition 13. (Abstract cone) $\mathcal{C} \subset E$ is called a *cone* if

1. $\mathcal{C}$ is convex,
2. $\lambda\mathcal{C} \subset \mathcal{C}$ for any $\lambda \geq 0$,
3. $\mathcal{C} \cap -\mathcal{C} = \{0\}$.

Using $\mathcal{C}$ we define a partial order on E

Definition 14. (Partial order) For any $x, y \in E$, we write $x \leq_{\mathcal{C}} y$ if $y - x \in \mathcal{C}$.

For clarity we will use $\leq$ instead of $\leq_{\mathcal{C}}$ if there is no confusion about the cone.

Definition 15. If $\mathcal{C}$ is a cone, we define the dual cone $\mathcal{C}^*$ by

$$\mathcal{C}^* := \{f \in E^* : \forall x \in \mathcal{C}, (f, x) \geq 0\}.$$

The set $\mathcal{C}^*$ is a cone if $\mathcal{C} - \mathcal{C}$ is dense and in particular if $\mathcal{C}$ has nonempty interior. We say that $x, y \in \mathcal{C}$ are *comparable and write* $x \sim y$ if there exist $\alpha, \beta > 0$ such that $\alpha x \leq y \leq \beta x$. This defines an equivalence relation. We say that $\mathcal{C}$ is *normal* if there exists $\gamma > 0$ such that

$$\forall x, y \in \mathcal{C}, 0 \leq x \leq y \Rightarrow \|x\| \leq \gamma \|y\|.$$

Definition 16. For any $x, y \in \mathcal{C}$ comparable, we define the Hilbert metric by

$$d_{\mathcal{C}}(x, y) = \log \frac{\beta_{\min}(x, y)}{\alpha_{\max}(x, y)}$$

where

$$\alpha_{\max}(x, y) = \sup \{\alpha > 0 : \alpha x \leq y\}$$

and

$$\beta_{\min}(x, y) = \inf \{\beta > 0 : y \leq \beta x\}$$

The Hilbert metric is a metric on the projective space of $\mathcal{C}$.

We say that $T : E \to E$ is *order-preserving* if $x \leq y \Rightarrow T(x) \leq T(y)$. If T is a linear operator (the only case we will consider here) this is equivalent to $T(\mathcal{C}) \subset \mathcal{C}$.

Remark 17. If T is order-preserving then T is non-expanding for the Hilbert metric. Indeed $\alpha x \leq y \leq \beta x$ implies $\alpha T(x) \leq T(y) \leq \beta T(x)$.

We introduce the projective diameter

$$\Delta(T) = \sup \{d_{\mathcal{C}}(T(x), T(y)) : x, y \in \mathcal{C}, \quad T(x) \sim T(y)\}$$

and the contracting ratio

$$\kappa(T) = \inf \{c > 0 : \forall x, y \; d_{\mathcal{C}}(T(x), T(y)) \leq c d_{\mathcal{C}}(x, y), \quad T(x) \sim T(y)\}$$

Here is the main theorem we will use :

Theorem 18. *(Birkhoff-Hopf [5, 22]) If T is order-preserving then*

$$\kappa(T) = \tanh\left(\frac{\Delta(T)}{4}\right).$$

The result has to be understood as follows: if the image of the cone of the order preserving operator is strictly inside the cone ($\Delta(T) < \infty$), then the operator is strictly contracting ($\kappa(T) < 1$) for the Hilbert metric.

3.2 Application to statistical physics

We now use the previous formalism to study the partition function and the marginals from statistical physics in the abstract framework of positive operators.

Definition 19. (Density function) Let $u \in \mathcal{C}^*$ and $v \in \mathcal{C}$, and let $X, Y, (X_i)_i$ be positive bounded operators. We define

- the partition function by
$$\mathcal{Z} = (u, T_N \dots T_0 v),$$
- the one-point density function by
$$\rho_{K_1}(X) = \frac{1}{\mathcal{Z}}(u, T_N \dots T_{K_1+1} X T_{K_1} \dots T_0 v),$$
- the pair correlation function by
$$\rho_{K_2,K_1}(Y, X) = \frac{1}{\mathcal{Z}}(u, T_N \dots T_{K_2+1} Y T_{K_2} \dots T_{K_1+1} X T_{K_1} \dots T_0 v),$$
- the k-point correlation function by
$$\rho_{K_k,\dots,K_2,K_1}(X_k, \dots, X_1) = \frac{1}{\mathcal{Z}}(u, T_N \dots T_{K_k+1} X_k T_{K_k} \dots T_{K_1+1} X_1 T_{K_1} \dots T_0 v).$$

The operators $X, Y, (X_i)$ should be thought of as test functions acting on the position of the K_i^{th} particle.

Remark 20. The simplest model that can be written in this formalism is the one-dimensional Ising model [31]. All the results stated above for Jellium can be easily adapted to the inhomogeneous one-dimensional Ising model.

We also think of Markov processes on a finite or compact set, in which case T_i is the transitive kernel from X_i to X_{i+1}.

3.2.1 Decay of correlations

The following theorem states the exponential decay of the correlation functions.

Theorem 21. *(Decay of correlations) Let $(T_i)_{i=1,\dots,N}$ be positive operators such that*
$$\Delta(T_i(C)) \leq M < \infty$$
for any i. Then there exist $c > 0$ which depends only on k, such that for $\min |K_{j+1} - K_j|$ *large enough, we have*

$$\left(1 - c\kappa^{\min_j |K_{j+1}-K_j|}\right) \prod_{i=1}^{k} \rho_{K_i}(X_i)$$

$$\leq \rho_{K_k,\dots,K_1}(X_k, \dots, X_1) \leq \left(1 + c\kappa^{\min_j |K_{j+1}-K_j|}\right) \prod_{i=1}^{k} \rho_{K_i}(X_i)$$

with $\kappa = \tanh(\frac{M}{4})$.

The decay of correlations is an important concept in statistical physics and it is ubiquitous in one-dimensional systems [32].

The proof of Theorem 21 is provided below in Section 3.4.

3.2.2 Regularity of the free energy

The second theorem states that the partition function depending on a parameter is smooth, if the transitive operators are smooth enough. In order to express the "regularity" of the operator in the framework of a cone and the Hilbert distance, we have to construct the following norm. The following result says that the distance is close to being a norm in the neighborhood of any point x_0.

Proposition 22. *Let $x_0 \in \mathcal{C}$. For any $\epsilon > 0$, there exists $r > 0$, a function f and a norm $\mathcal{N}$ defined on the projective space, such that d can be written as follows*

$$d(x, y) = f(x, y)\mathcal{N}(y - x)$$

for all x, y such that $d(x, x_0) < r$ and $d(y, x_0) < r$, with $|f(x, y) - 1| < \epsilon$.

We can now state our second main result

Theorem 23. *Let $\mathcal{C}$ be a cone and let $T_i(\beta)$ be a family of smooth bounded operators for β in the neighborhood $[\beta_0 - \delta, \beta_0 + \delta]$ of β_0, which are contracting of parameter $\kappa < 1$, uniformly in β and i. For all i, we denote by $\mathcal{N}_i$ the norms defined in Proposition 22 around $x_i = \prod_{j=0}^{i-1} T_j(\beta)b$. Assume that the derivatives in β of the operator are uniformly bounded for theses norm, that is,*

$$\exists C', \forall i, \quad \left\| \frac{d^k T_i}{d\beta^k} \right\|_{\mathcal{N}_i \to \mathcal{N}_{i+1}} \leq C'_k$$

for some constant C'_k independent of i and of $\beta \in [\beta_0 - \delta, \beta_0 + \delta]$. Then

$$f_N(\beta) = \frac{1}{N} \log \left\langle a, \prod_{i=0}^{N-1} T_i(\beta) b \right\rangle$$

is uniformly smooth, meaning there is a constant C which depends only on κ and (M_k) such that :

$$\left|\frac{d^n f_N(\beta)}{d\beta^n}\right| \leq C(\kappa, (M_k)_{k\leq n})$$

where $M_k = \sup_{i,\beta\in[\beta_0-\delta,\beta_0-\delta]} \|\frac{d^k T_i}{d\beta^k}\|_{\mathcal{N}_i\to\mathcal{N}_{i+1}}$.

Moreover if the following limit exists

$$f(\beta) = \lim_{k\to\infty} \frac{1}{N_k} \log \left\langle a, \prod_{i=0}^{N_k-1} T_i(\beta) b \right\rangle$$

for a sequence $N_k \to \infty$*, then it is smooth in a neighborhood of* β_0*:*

$$\left|\frac{d^n f(\beta)}{d\beta^n}\right| \leq C(\kappa, (M_k)_{k\leq n}).$$

If the positive operator appears to be uniformly analytic for the constructed norm then the free energy is analytic. More precisely we have the following theorem

Theorem 24. *With the same assumptions as in Theorem 23*, if there exists $r \geq 0$ such that

$$\frac{\|\partial_\beta^n T_i\|_{\mathcal{N}_i\to\mathcal{N}_{i+1}}}{n!} \leq r^n$$

for all n*, then* f *is real analytic around* β *with radius of convergence at least equal to* $(1-\kappa)/r$*.*

The two theorems of this section are proved later in Section 3.5.

3.2.3 Central Limit Theorem

We consider the particular case where the space is $L^1(\Lambda)$, with Λ a measurable set and the cone is $\mathcal{C} = \{f \in L^1(\Lambda) : f \geq 0\}$. We construct the canonical random process y_i as follows. Let $A_1, \cdots, A_N \subset \Lambda$, and take as test functions $1_{A_1}, 1_{A_2}..., 1_{A_N}$. Then we define

$$\mathbb{P}(y_1 \in A_1, y_2 \in A_2, ..., y_N \in A_N) = \rho_{1,\cdots,N}(1_{A_1}, ..., 1_{A_N}) = \frac{1}{Z}\left\langle \prod_{i=1}^{N} T_i 1_{A_i} \right\rangle. \tag{4}$$

The decay of correlation in Theorem 21 is the mixing property of the process (y_i).

The Central Limit Theorem has been proved for a urge number of random processes like martingales [19], Markov processes[20, 18] or random products of matrices [27]. One of the classical proofs of the central limit theorem uses the regularity of the Laplace transform, this is what we adapt here.

Theorem 25. *(Central Limit) Let $h_i : \Lambda \to \mathbb{R}$ be such that $\mathbb{E}(\exp(h_i(y_i))) < \infty$ where the mean is on the probability (4). Let $T_i(\beta) = e^{\beta h_i(y_i)} T_i$. If the $T_i(\beta)$ satisfy the assumptions of Theorem 23, then we have*

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{1}{\sqrt{N}} \sum (h_i(y_i) - \mathbb{E}(h_i(y_i))) \geq x \right) - \mathbb{P}\big(\mathcal{N}(0,\sigma^2) \geq x\big) \right| = O\left(\frac{1}{\sqrt{N}}\right)$$

where σ^2 is the second derivative of the free energy

$$\sigma^2 = \frac{d^2}{d\beta^2} \left[\frac{1}{N} \log \left\langle a, \prod_{i=0}^{N-1} T_i(\beta) b \right\rangle \right].$$

This theorem is proved in Section 3.6.

3.3 Rank-one operator approximation

One of the first historical applications of the Birkhoff-Hopf Theorem was in population demography [11]. An age structure diagram f evolves due to birth and death, with death and birth rates not constant in time and one can calculate its time evolution. It appears that even if f does not converge to an equilibrium, the long time evolution is independent of the initial age structure $f(0)$. Namely this is a weak ergodicity property: if f_1 and f_2 are two solutions of the evolution with different initial data, $\|f_1(t)/\|f_1(t)\| - f_2(t)/\|f_2(t)\|\| \to 0$.

In this section, we formulate weak ergodicity in term of a rank-one operator approximation and we give a construction and an estimate of such an approximation in case where several contracting operators are composed one after another.

3.3.1 The cone of order preserving operators

We state here some simple results about the set of order preserving operators.

Lemma 26. *Let $\mathcal{C}$ with $\mathcal{C} - \mathcal{C}$ dense. The set of corresponding order-preserving operator is a cone.*

We denote by $\mathcal{P}_\mathcal{C}$ this cone and only $\mathcal{P}$ if there is no confusion.

Proof. We check every point of the definition.

1. If A, B are order preserving operator then $A + B$ is an order preserving operator. Indeed $(A + B)(x) \in \mathcal{C}$ for all $x \in \mathcal{C}$.

2. The set of order preserving operator is invariant by product of strictly positive scalars.

3. Let $A \in \mathcal{P} \cap -\mathcal{P}$, then $(f, Ax) = 0$ for all $x \in \mathcal{C}$ and all $f \in \mathcal{C}^*$. Therefore $(f_1 - f_2, A(x_1 - x_2)) = 0$ for all $x_1, x_2 \in \mathcal{C}$ and all $f_1, f_2 \in \mathcal{C}^*$. Therefore $A = 0$ since $\mathcal{C}^* - \mathcal{C}^*$ and $\mathcal{C} - \mathcal{C}$ are dense.

□

Example 27. One can think of $\mathcal{C}$ the positive vectors in $\mathbb{R}^n$ and the set of matrices $\mathcal{M}_n(\mathbb{R})$ with positive coefficients.

We have the following order on the set of operators :

$$B \geq_{\mathcal{P}} A \Leftrightarrow (B-A)(\mathcal{C}) \subset \mathcal{C}$$

and the corresponding Hilbert distance

$$d_{\mathcal{P}}(A,B) = \min\Big(\log\big(\frac{\beta}{\alpha}\big) : \alpha A \leq B \leq \beta A\Big).$$

Remark 28. If $A \leq_{\mathcal{P}} B$ and $C \leq_{\mathcal{P}} D$ then $AC \leq_{\mathcal{P}} BD$. Indeed $(B-A)C(\mathcal{C}) \subset (B-A)(\mathcal{C}) \subset \mathcal{C}$, and $B(D-C)(\mathcal{C}) \subset B(\mathcal{C}) \subset \mathcal{C}$. Therefore $BD \geq_{\mathcal{P}} BC \geq_{\mathcal{P}} AC$.

Unfortunately that T is contracting does not imply that $\tilde{T} : A \to TA$ is contracting as well. One can take for example:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

in which case $d_{\mathcal{P}}(A,B) = \log(\frac{2}{(1/2)}) = \log(4)$ and $d_{\mathcal{P}}(TA,TB) = \log(4)$ as well.

Lemma 29. *Let A, B, C, D be increasing operators. Then*

$$d_{\mathcal{P}}(AB, CD) \leq d_{\mathcal{P}}(A,C) + d_{\mathcal{P}}(B,D).$$

Proof. Let $\alpha_1, \beta_1, \alpha_2, \beta_2$ such that $\alpha_1 A \leq C \leq \beta_1 A$, $\alpha_2 B \leq D \leq \beta_2 B$ and $\log\big(\frac{\beta_1}{\alpha_1}\big) - d_{\mathcal{P}}(A,C) \leq \epsilon$, $\log\big(\frac{\beta_2}{\alpha_2}\big) - d_{\mathcal{P}}(B,D) \leq \epsilon$. Then $\alpha_1\alpha_2 AC \leq BD \leq \beta_1\beta_2 AC$ and we have $d_{\mathcal{P}}(AB,CD) \leq \log\big(\frac{\beta_1\beta_2}{\alpha_1\alpha_2}\big) \leq d_{\mathcal{P}}(A,C) + d_{\mathcal{P}}(B,D) + 2\epsilon$. □

Now we construct a rank-one operator $L = z \cdot l$, with a vector $z \in E$, and a linear form $l \in E^*$ to approximate a contracting function T. It is natural to choose $z \in T(C)$. We construct l in the following subsection.

3.3.2 Rank-one operator construction

We construct here the rank-one operator close to a contracting operator.

Lemma 30. *Let $a : C \to \mathbb{R}_+$ and $b : C \to \mathbb{R}_+$ be such that*

1. *there exist $M_1, M_2 < \infty$, for all x, $a(x) \leq M_1\|x\|$ and $b(x) \leq M_2\|x\|$,*
2. *for all $\lambda \geq 0$ and $x \in C$, $a(\lambda x) = \lambda a(x)$ and $b(\lambda x) = \lambda b(x)$,*
3. *for all $x \in C$ $a(x) \leq b(x)$,*
4. *for all $x, y \in C$ $a(x+y) \geq a(x) + a(y)$ and $b(x+y) \leq b(x) + b(y)$.*

Then there exists a linear form $l \in C^*$ such that, for any $x \in C$,

$$a(x) \leq l(x) \leq b(x).$$

Proof. We check that b is a convex function,

$$b(tx + (1-t)y) \leq b(tx) + b((1-t)y) = tb(x) + (1-t)b(y),$$

and that a is a concave function,

$$a(tx + (1-t)y) \geq a(tx) + a((1-t)y) = ta(x) + (1-t)a(y).$$

Let us define two sets : $B = \{(x,s) \in C \times \mathbb{R} : b(x) \leq s\}$ and $A = \{(x,s) \in C \times \mathbb{R} : a(x) > s\}$. Then $A \cap B = \emptyset$, A and B are convex. Because of the Hahn-Banach separation theorem, there exists $l \neq 0$ a linear form on $E \times \mathbb{R}$ such that for all $(x,s) \in B$, $l(x,s) \geq 0$ and all $(x,s) \in A$, $l(x,s) \leq 0$. We have $l(x,s) = l_1(x) + \alpha s$ with $l_1 \in E^*$ and $\alpha \in \mathbb{R}$.

We then prove that $\alpha > 0$. For any $s_0 > 0$ we have $(0, s_0) \in B$, $(0, -s_0) \in A$, $\alpha s_0 \geq -\alpha s_0$ and as a conclusion $\alpha \geq 0$. If $\alpha = 0$, because for any $x \in C$ there exist s_1 and s_2 such that $(x, s_1) \in B$ and $(x, s_2) \in A$, we have $0 \geq l(x, s_2) = l_1(x) = l(x, s_1) \geq 0$ and therefore $l_1(x) = 0$. Let $x_0 \in \mathring{C}$, and $V_\epsilon(x_0) \subset C$ a small ball with center x_0 and radius ϵ. Then for all y with $\|y\| < \epsilon$, we have $l_1(y) = l_1(y + x_0) = 0$. As a conclusion $l = l_1 = 0$ which is absurd, so $\alpha \neq 0$.

Let $l_0 = -\frac{l_1}{\alpha}$. Since $(x, b(x)) \in B$, $-l_0(x) + b(x) \geq 0$ we have $l_0(x) \leq b(x)$. Moreover for $\epsilon > 0$, $(x, a(x) - \epsilon) \in A$ $-l_0(x) + a(x) - \epsilon \leq 0$ and therefore $l_0(x) \geq a(x)$. □

Corollary 31. *There exists a rank one operator $L_T = z \cdot l$ with $z \in \mathcal{C}$ and $l \in \mathcal{C}^*$ such that $d_{\mathcal{P}}(T, L_T) \leq 2\Delta(T)$.*

Proof. Let $z = T(y_0) \in T(\mathcal{C})$ and define a and b as follows: for any $x \in \mathcal{C}$

$$a(x) =_{def} \max\{\alpha : \alpha T(y_0) \leq_{\mathcal{C}} T(x)\}$$

and

$$b(x) =_{def} \min\{\beta : T(x) \leq_{\mathcal{C}} \beta T(y_0)\}.$$

It is possible to check the hypothesis of Lemma 30. Indeed we have that

$$\begin{cases} \alpha_1 T(y_0) \leq_{\mathcal{C}} T(x_1) \leq_{\mathcal{C}} \beta_1 T(y_0), \\ \alpha_2 T(y_0) \leq_{\mathcal{C}} T(x_2) \leq_{\mathcal{C}} \beta_2 T(y_0) \end{cases}$$

implies

$$(\alpha_1 + \alpha_2) T(y_0) \leq_{\mathcal{C}} T(x_1) + T(x_2) \leq_{\mathcal{C}} (\beta_1 + \beta_2) T(y_0)$$

and therefore $a(x_1 + x_2) \geq a(x_1) + a(x_2)$ and $b(x_1 + x_2) \leq b(x_1) + b(x_2)$. We also have

$$a(\lambda x) = \lambda a(x) \text{ and } b(\lambda x) = \lambda b(x)$$

for all $\lambda \geq 0$ and $x \in C$. We can then apply Lemma 30: there exists a linear form l with $a(x) \leq l(x) \leq b(x)$. Moreover $\log \frac{b(x)}{a(x)} \leq \Delta(T)$ for all $x \in \mathcal{C}$. We then have for all x $\frac{l(x)}{a(x)} \leq e^{-\Delta(T)}$ and $\frac{b(x)}{l(x)} \leq e^{\Delta(T)}$ and therefore

$$e^{-\Delta(T)}T(x) \leq_{\mathcal{C}} T(y_0) \cdot l(x) \leq_{\mathcal{C}} e^{\Delta(T)}T(x).$$

As a conclusion $d_{\mathcal{P}}(T(y_0) \cdot l, T) \leq \log(e^{2\Delta(T)}) \leq 2\Delta(T)$. □

Corollary 32. *Let $(T_i)_{i=0,...,N}$ be positive operators. If*

1. $\Delta(T_0(\mathcal{C})) \leq R < \infty$,
2. T_i $i = 1, \dots, N$ *are uniformly contracting of parameter* $\kappa < 1$,

then there exists a linear form l, $z_0 \in \mathcal{C}$, $\|z_0\| = 1$ and $l \in \mathcal{C}^*$ such that

$$d_{\mathcal{P}}\big((T_N \dots T_0), z_0 \cdot l\big) \leq 2\kappa^N R.$$

Proof. We have $\Delta(T_N \dots T_0) \leq \kappa^N R$ and the result follows from the previous corollary. □

3.4 Decay of correlation function

Here we prove Theorem 21. The idea is to replace the product of contracting operator between the k points of measure by a rank-one operator. We will do so for $k = 2$ and for $k > 2$ this will be exactly the same. More precisely we will prove the following

Theorem 33. *(Theorem 21 in the case $k = 2$) Let $(T_i)_{i=1,...,N}$ be positive operators such that $\Delta(T_i(C)) \leq M < \infty$ for any i, and let $K_1, K_2 \in \mathbb{N}$ be such that $1 \leq K_1 \leq K_2 \leq N$. Let $u, v \in C$, and X, Y be two positive operators.Then:*

$$\begin{aligned} &e^{-8R(\kappa^{K_1}+\kappa^{K_2-K_1}+\kappa^{N-K_2})}\rho_{K_1}(X)\rho_{K_2}(Y) \\ &\qquad \leq \rho_{K_2,K_1}(Y,X) \leq e^{8R(\kappa^{K_1}+\kappa^{K_2-K_1}+\kappa^{N-K_2})}\rho_{K_1}(X)\rho_{K_2}(Y). \end{aligned}$$

One can use this theorem for K_1,$K_2 - K_1$, $N - K_2$ large. In this case, the Taylor expansion of e^x gives

$$|\rho_{K_1}(X)\rho_{K_2}(Y) - \rho_{K_1,K_2}(X,Y)| \leq 16R(\kappa^{K_1} + \kappa^{K_2-K_1} + \kappa^{N-K_2})\|X\|\|Y\|,$$

which decays exponentially.

Proof. Let us introduce $L_{K_10} = z_{K_10}l_{K_10}$, $L_{K_2K_1} = z_{K_2K_1}l_{K_2K_1}$ and $L_{NK_2} = z_{NK_2}l_{NK_2}$ which are rank-one operators such that

$$\begin{cases} d_{\mathcal{P}}\big((T_{K_1} \dots T_0), L_{K_10}\big) \leq 2\kappa^{K_1}R, \\ d_{\mathcal{P}}\big((T_{K_2} \dots T_{K_1}), L_{K_2K_1}\big) \leq 2\kappa^{K_2-K_1}R, \\ d_{\mathcal{P}}\big((T_N \dots T_{K_2}), L_{NK_2}\big) \leq 2\kappa^{N-K_2}R. \end{cases}$$

We then use Proposition 29, to obtain the inequality for the partition function,

$$d_{\mathcal{P}}(T_N \ldots T_0, L_{NK_2} L_{K_2K_1} L_{K_10}) \leq 2R(\kappa^{N-K_2} + \kappa^{K_2-K_1} + \kappa^{K_1}),$$

for the density function

$$d_{\mathcal{P}}(T_N \ldots T_{K_1+1} X T_{K_1} \ldots T_0, L_{NK_2} L_{K_2K_1} X L_{K_10}) \leq 2R(\kappa^{N-K_2}+\kappa^{K_2-K_1}+\kappa^{K_1})$$
$$d_{\mathcal{P}}(T_N \ldots T_{K_2+1} Y T_{K_2} \ldots T_0, L_{NK_2} Y L_{K_2K_1} L_{K_10}) \leq 2R(\kappa^{N-K_2}+\kappa^{K_2-K_1}+\kappa^{K_1}),$$

and the pair correlation function

$$\begin{aligned} & d_{\mathcal{P}}(T_N \ldots T_{K_2+1} Y T_{K_2} \ldots T_{K_1+1} X T_{K_1} \ldots T_0, L_{NK_2} Y L_{K_2K_1} X L_{K_10}) \\ & \qquad \leq 2R(\kappa^{N-K_2} + \kappa^{K_2-K_1} + \kappa^{K_1}). \end{aligned}$$

Moreover we have

$$\begin{aligned} & (u, L_{NK_2} L_{K_2K_1} L_{K_10} v) \cdot (u, L_{NK_2} Y L_{K_2K_1} X L_{K_10} v) \\ & \quad = (u, z_{NK_2})(l_{NK_2} z_{K_2K_1})(l_{K_2K_1} z_{K_10})(l_{K_10} v)(u, z_{NK_2})(l_{NK_2} Y(z_{K_2K_1})) \\ & \qquad (l_{K_2K_1} X(z_{K_10}))(l_{K_10} v) \\ & \quad = (u, z_{NK_2})(l_{NK_2} z_{K_2K_1})(l_{K_2K_1} X(z_{K_10}))(l_{K_10} v)(u, z_{NK_2})(l_{NK_2} Y(z_{K_2K_1})) \\ & \qquad (l_{K_2K_1} z_{K_10})(l_{K_10} v) \\ & \quad = (u, L_{NK_2} L_{K_2K_1} X L_{K_10} v) \cdot (u, L_{NK_2} Y L_{K_2K_1} L_{K_10} v) \end{aligned}$$

and this allows us to conclude that

$$\begin{aligned} & Z^2 \rho_{K_1}(X) \rho_{K_2}(Y) \\ & \quad \leq (u, L_{NK_2} L_{K_2K_1} X L_{K_10} v) \cdot (u, L_{NK_2} Y L_{K_2K_1} L_{K_10} v) e^{4R(\kappa^{N-K_2}+\kappa^{K_2-K_1}+\kappa^{K_1})} \\ & \quad \leq (u, T_N \ldots T_{K_2+1} X T_{K_2} \ldots T_{K_1+1} X T_{K_1} \ldots T_0 v) \cdot (u, T_N \ldots T_0 v) \\ & \qquad \times e^{8R(\kappa^{N-K_2}+\kappa^{K_2-K_1}+\kappa^{K_1})} \\ & \quad \leq Z^2 \rho_{K_2,K_1}(Y, X) e^{8R(\kappa^{N-K_2}+\kappa^{K_2-K_1}+\kappa^{K_1})}. \end{aligned}$$

Finally, we have

$$Z^2 \rho_{K_1}(X) \rho_{K_2}(Y) \geq Z^2 \rho_{K_2,K_1}(Y, X) e^{-8R(\kappa^{N-K_2}+\kappa^{K_2-K_1}+\kappa^{K_1})}.$$

□

The proof of the decay of the cluster correlation is the same. One should just replace X_i by $X_i = T_{i+l} Y_{i,l-1} \cdots T_{i+1} Y_{i,2} T_i Y_{i,1}$, which are positive operators.

3.5 Smoothness of the free energy

In this section, we prove Proposition 22 and Theorem 23.

3.5.1 Proof of Proposition 22

Proof. Let H a hyperplane such that $E = \mathrm{Vect}(\{x_0\}, H)$. The projective space is locally isomorph to H. Let B be the convex set containing all the $s \in H$ for which there exist $(\alpha_+, \alpha_-, \beta_-, \beta_+)$ satisfying $\alpha_+ x_0 \le x_0 + s \le \beta_+ x_0$, with $\beta_+ - \alpha_+ \le r$, and $\alpha_- x_0 \le x_0 - s \le \beta_- x_0$, with $\beta_- - \alpha_- \le r$, for some r small enough. This set is symmetric with respect to the transformation $s \to -s$. Therefore, it is the ball of the norm $\|s\| = r \cdot \inf(\lambda \in \mathbb{R}, \frac{s}{\lambda} \in B))$. Let us check that this norm is close to the distance. Let $s_1, s_2 \in H$ be such that $d(x_0+s_1, x_0) < r$ and $d(x_0+s_2, x_0) < r$. We have then $\alpha_1 x_0 \le x_0+s_1 \le \beta_1 x_0$ and $\alpha_2 x_0 \le x_0+s_2 \le \beta_2 x_0$ and because r is very small, we can write $\alpha_1 = 1+\delta\alpha_1$, $\alpha_2 = 1+\delta\alpha_2$, $\beta_1 = 1+\delta\beta_1$, $\beta_2 = 1+\delta\beta_2$. At first order we have $d(x_0+s_1, x_0) = \delta\beta_1 - \delta\alpha_1 + o(|\delta\beta_1|, |\delta\alpha_1|)$ and $d(x_0+s_2, x_0) = \delta\beta_2 - \delta\alpha_2 + o(|\delta\beta_2|, |\delta\alpha_2|)$.

We now check that $d(x_0+s_1, x_0) = (1+O(r))\|s_1\|$. First we have

$$d(x_0+s_1, x_0) \ge (1+O(r))\|s_1\|.$$

Indeed, for any $\lambda \in \mathbb{R}$, we have $\lambda(1+\delta\alpha_1)x_0 + (1-\lambda)x_0 \le x_0 + \lambda s \le \lambda(1+\delta\beta_1)x_0 + (1-\lambda)x_0$ and $x_0 + \lambda(\delta\alpha_1)x_0 \le x_0 + \lambda s \le x_0 + \lambda\delta\beta_1 x_0$. With $\lambda = \frac{r}{\delta\beta_1 - \delta\alpha_1}$, we obtain

$$x_0 + \frac{r}{\delta\beta_1 - \delta\alpha_1}(\delta\alpha_1)x_0 \le x_0 + \frac{r}{\delta\beta_1 - \delta\alpha_1} s \le x_0 + \frac{r}{\delta\beta_1 - \delta\alpha_1}\delta\beta_1 x_0$$

Therefore

$$\|s\| \le \delta\beta_1 - \delta\alpha_1 = (1+O(r))d(x_0, x_0+s).$$

Then we claim that

$$d(x_0+s_1, x_0) \le (1+O(r))\|s_1\|.$$

Indeed let λ be such that for any α, β $\alpha x_0 \le x_0 + \frac{s}{\lambda} \le \beta x_0 \Rightarrow \beta - \alpha \ge r$. Then for any α, β $\lambda\alpha x_0 + (1-\lambda)x_0 \le x_0 + s \le \lambda\beta x_0 + (1-\lambda)x_0 \Rightarrow \beta - \alpha \ge r$. Therefore

$$\begin{aligned} d(x_0, x_0+s) \le \log\frac{1-\lambda+\lambda\beta}{1-\lambda+\lambda\alpha} &= \log\frac{1+\lambda\delta\beta}{1+\lambda\delta\alpha} \\ &= (\lambda(\delta\beta - \delta\alpha))(1+O(r)) \le \|s\|(1+O(r)). \end{aligned}$$

We finally check that $d(x_0+s_1, x_0+s_2) = (1+O(r))d(x_0+s_1-s_3, x_0+s_2-s_3)$ for any $s_1, s_2, s_3 \in B$. We have, $\alpha_3 x_0 \le x_0+s_3 \le \beta_3 x_0$, and $d(x_0+s_1, x_0+s_2) = (\delta\beta - \delta\alpha)(1+O(r))$ with $(1+\delta\alpha)(x_0+s_1) \le x_0+s_2 \le (1+\delta\beta)(x_0+s_1)$. Then

$$(1+\delta\alpha+O(r)\delta\alpha)(x_0+s_1+s_3) \le (1+\delta\alpha)(x_0+s_1+s_3) - \delta\alpha s_3 \le x_0+s_2+s_3$$

and

$$x_0+s_2+s_3 \le (1+\delta\beta)(x_0+s_1+s_3) - \delta\alpha s_3 \le (1+\delta\beta+O(r))(x_0+s_1+s_3).$$

We conclude that $d(x_0+s_1, x_0+s_2) = \|s_2 - s_1\|(1+O(r))$. □

Let $\mathcal{C}$ be the cone of positive vectors in $\mathbb{R}^n$ and let $x_0 = (x^1, \cdots, x^n)$ and $H = \{s : \sum_i s^i = 0\}$. Then in a neighborhood of x_0, we have

$$\alpha x_0 \leq x_0 + s \leq \beta x_0$$

with

$$\alpha = \max_{\alpha} \alpha x^i \leq x^i + s^i = 1 + \min \frac{s^i}{x^i}$$

and

$$\beta = \min_{\beta} \beta x^i \geq x^i + s^i = 1 + \max \frac{s^i}{x^i}.$$

In addition

$$d(x_0, x_0 + s) = \log \frac{1 + \max \frac{s^i}{x^i}}{1 + \min \frac{s^i}{x^i}} \approx \max \frac{s^i}{x} - \min \frac{s^i}{x} = \max \frac{s^i}{x^i} + \max \frac{-s^i}{x^i}.$$

Finally the constructed norm is then:

$$\|s_1 - s_2\| = \max \frac{s_1^i - s_2^i}{x^i} + \max \frac{s_2^i - s_2^i}{x^i}.$$

In order to prove Theorem 23 we will need the following lemma.

Lemma 34. *Let E_n be Banach spaces with norms $\|\|_n$. Consider the functions $u_n : \mathbb{R} \to E_n$ iteratively defined by*

$$u_{n+1}(s) = g_n(s, u_n(s)),$$

with $g_n(s, 0) = 0$ which are assumed to be uniformly contracting, $\|\partial_2 g_n\|_{\|\|_n \to \|\|_{n+1}} \leq \kappa$ with $\kappa < 1$. If the g_n are uniformly C^k then the u_n are uniformly C^k.

Proof. We prove by induction that there exists constant a C_k such that for all n, $\|\frac{d^k}{ds^k} u_n\|_n \leq C_k$. Computing the derivative gives

$$\frac{d^k}{ds^k} u_{n+1} = \partial_2 g_n \cdot \frac{d^k}{ds^k} u_n + Q\Big(g_n, (\partial_1^r \partial_2^p g_n), (\frac{d^i}{ds^i} u_n)_{i<k}\Big)$$

where Q is a polynomial involving lower order derivatives of u_n and the derivatives of g_n. Because of the induction hypothesis, there exists C_{k-1} such that for all n and all $i < k$, $\|\frac{d^i}{ds^i} u_n\| \leq C_{k-1}$. Therefore Q can be uniformly bounded by a constant $\tilde{C}_k$ which depends only on $\sup_{n \in \mathbb{N}} \|\partial_1^r \partial_2^p g_n\|$ and C_{k-1}. We have therefore

$$\|\frac{d^k}{ds^k} u_{n+1}\|_{n+1} \leq \kappa \|\frac{d^k}{ds^k} u_n\|_n + \tilde{C}_k$$

and we can then set

$$C_k = \frac{1}{1 - \kappa} \tilde{C}_k. \tag{5}$$

We can now conclude because if g is contracting for d then it is contracting for $\|\|$. □

Example 35. Consider $T(\beta) = \begin{pmatrix} \beta & \epsilon \\ \epsilon & 1 \end{pmatrix}$, with largest eigenvalue

$$\lambda(\beta) = \frac{(\beta+1) + \sqrt{(\beta-1)^2 + 4\epsilon^2}}{2}$$

and $\log(\lambda(\beta))$ for β around 1. In the usual positive cone,

$$\Delta(T(\beta)(\mathcal{C})) = d_{\mathcal{C}}\left(\begin{pmatrix} 1 \\ \epsilon \end{pmatrix}, \begin{pmatrix} \epsilon \\ 1 \end{pmatrix}\right) = |2\log(\epsilon)|.$$

The Birkhoff-Hopf theorem gives $\kappa = \tanh(\log(\epsilon)/2) \approx 1 - 2\epsilon$. Around the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the norm $\mathcal{N}$ is equal to the norm $\|.\|_\infty$ (see example 3.5.1). The iterative formula (5) gives a constant behave like $C_k \approx (2\epsilon)^{(1-k)}$ and this is what we get with the exact calculation of $\frac{d}{d\beta^k}[\log(\lambda(\beta))]$.

Remark 36. If T is contracting for the distance d, then T is locally contracting for $\mathcal{N}$.

We can now finish the proof of Theorem 23.

Proof. [Theorem 23]We denote

$$u_n(\beta) = \frac{\prod_{i=0}^{n-1} T_i(\beta) b}{\| \prod_{i=0}^{n-1} T_i(\beta)\|}$$

and we decompose the log of the product as

$$f_N(\beta) = \frac{1}{N}\log\left\langle a, \prod_{i=0}^{N-1} T_i(\beta)b\right\rangle = \frac{1}{N}\log\langle a, u_N(\beta)\rangle + \frac{1}{N}\sum \log(\|T_i(\beta)u_i(\beta)\|).$$

Because the T_i are smooth we only have to make sure that the u_i are smooth as well. This follows from the previous lemma. The function $u_i(\beta)$ is smooth for the constructed norm $\|\|_i$. But because the cone is normal,

$$\alpha x \le y \le \beta x \Rightarrow \|y - \frac{\beta+\alpha}{2}x\| \le \frac{\beta-\alpha}{2}\|x\|,$$

we have then that $\|x\| \cdot \|x-y\|_i \ge \|x-y\|$ and we conclude because $\|u_n(\beta)\| = 1$. ☐

Proposition 37. *Let $g_1, g_2, \cdots, g_n \cdots$, be analytic functions such that for any k, $g_k(x) = \sum_i b_{k,n}x^n$ with $|b_{k,n}| \le r^n$ and $|b_{k,0}| \le 1$. Let $f_0 = g_0$ and $f_{k+1} = (1 + \kappa g_{k+1} f_k)$. Then for any k,*

$$f_k = \sum c_{k,n}x^n$$

with $c_{k,n} \le d_n$ where d_n are the coefficient of the Taylor expansion of $\frac{1-rx}{(1-\kappa)-rx}$. In particular, if f_n admits a limit f_∞, then f_∞ is analytic.

Proof. We can assume $b_{k,n} = r^n$ for any k, n. Indeed another configuration would give a smaller $c_{k,n}$. We expand f_k and have: $f_k = \sum_{i=0}^{k} \left(\frac{\kappa}{1-rx}\right)^i$ whose coefficients are then smaller than those of $\sum_{i=0}^{\infty} \left(\frac{\kappa}{1-rx}\right)^i = \frac{1-rx}{(1-\kappa)-rx}$. □

Corollary 38. *Let g_n and u_n be as in Proposition 34. Suppose that there exists $r \geq 0$ such that $\frac{\|\partial_s^i g\|}{i!} \leq r^i$ for all i, then u_n are analytic with coefficients of its Taylor series bounded by that of $\frac{1-rx}{(1-\kappa)-rx}$. In particular if u_n admits a limit then it is analytic with convergence radius $\frac{1-\kappa}{r}$.*

Proof. This follows from the fact that

$$u_n(s) - u_n(0) = g_n(s, u_{n-1}(s) - u_{n-1}(0)) + g_n(s, u_{n-1}(0)) - u_n(0).$$

□

3.6 Proof of Theorem 25

We now prove the central limit theorem 25 from the regularity of the Laplace transform.

Proof. By Theorem 24 $f(\alpha)$ is smooth with $\partial_\alpha f|_{\alpha=0} = \gamma\sqrt{N} \leq C\sqrt{N}$, $\partial_\alpha^2[f - \gamma\sqrt{N}\alpha]|_{\alpha=0} = \sigma^2 \leq C$ and $\partial_\alpha^3[f - \gamma\sqrt{N}\alpha] \leq \frac{C}{\sqrt{N}}$. Then $(f(\alpha) - \gamma\alpha) = 1 + \frac{(\sigma\alpha)^2}{2} + O(\frac{1}{\sqrt{N}})$. Therefore the Laplace transform is close to the one of a Gaussian and we can conclude with the usual Berry Essen inequality. □

4 Proofs for the Jellium model

4.1 Proof for the classical Jellium model

We first write the partition function in the form of products of operators. Recall that $U_i(s) = -2q_i \int_{\tilde{x}_i}^{\tilde{x}_i + s} (y - \tilde{x}_i)\rho(y)dy$ with $\tilde{x}_i$ the equilibrium position of the particle i. We note $\delta = \frac{1}{2}\min(|\tilde{a}_i - \tilde{a}_{i+1}|)$.

Definition 39. (Iterative operator) Let T_i be the operator defined for any function f in L^1 or L^∞ by

$$T_i f(x) = \int_{s=x-\tilde{x}_{i+1}+\tilde{x}_i}^{\infty} e^{-\beta U(s)} f(s) ds.$$

In particular, we can rewrite the partition function as

$$\mathcal{Z}_N(\beta) = e^{-\beta E(\tilde{x}_1,...,\tilde{x}_N)} \left\langle 1_{x_N < L - \tilde{x}_N}, \left(\prod_{i=0}^{N-1} T_i\right) 1_{x_1 > -L - \tilde{x}_1} \right\rangle$$

We are then in the setting of Section 3.4.

4.1.1 Construction of a uniform invariant cone

We first notice that we cannot directly apply the Birkhoff-Hopf Theorem with the cone of positive functions $\mathcal{C}_0$. Indeed we have the

Remark 40. For any T_i, $\Delta_{\mathcal{C}_0}(T_i) = \infty$. For example $\mathrm{Supp}[T_i(1_{[0,1]})] = (-\infty; 1 + (\tilde{x}_{i+1} - \tilde{x}_i)]$ and $\mathrm{Supp}(T_i(1_{[2,3]})) = (-\infty, 3 + (\tilde{x}_{i+1} - \tilde{x}_i)]$ and then we have for any $\alpha > 0$ and $1+(\tilde{x}_{i+1}-\tilde{x}_i) < t < 3+(\tilde{x}_{i+1}-\tilde{x}_i)$, $\big(T_i(1_{[0,1]})-\alpha T_i(1_{[2,3]})\big)(t) < 0$ so $\alpha_{\min} = 0$.

The solution is to construct another cone. If we were restricted to a bounded interval, then the simplest solution would be to consider finite products of T_i, instead of one by one. The kernel of $\prod_{i=n}^{n+k-1} T_i$ is strictly positive on $\{(x,y), y \geq x-2n\delta\}$, and therefore with n such that $2n\delta > 2A$, the kernel is strictly positive. $\prod T_i$ are then contracting for the cone $\{f \geq 0\}$.

In our case, because of the multiplication by $e^{-U_i(s)}$, we will be able to neglect the influence of $fe^{-U_i(s)}$ outside $(-A, A)$. We choose A such that

$$\int_A^\infty e^{-U_i(s)} ds \leq \frac{\delta}{2} e^{-U_i(A)}$$

(for example, because of $\frac{d}{ds}U(s) \geq qms$, we can choose $A = \frac{2}{\delta qm}$). In Proposition 41 we will define a cone such that f can be slightly negative for $\{x : x \geq A\}$. We also make it so that T_i are contracting and not only $\prod_k^{k+n} T_i$. The price to pay is more restrictions. Intuitively, it is how $\prod T_i f$ looks like for $f \geq 0$.

Let us divide the interval $[-A, A]$ in small intervals with $I_k = [k\delta/2, (k+1)\delta/2]$ with $k \in \mathbb{Z}$ and $-2\frac{A}{\delta} - 1 = k_{min} \leq k \leq k_{max} = 2\frac{A}{\delta} + 1$. We suppose $\frac{A}{\delta} \in \mathbb{N}$ to simplify the notation.

Proposition 41. *There exist $(\epsilon_k)_{-2\frac{A}{\delta} \leq k \leq 2\frac{A}{\delta}}$ such that the cone $\mathcal{C}$ defined by*

$$f \in \mathcal{C} \Leftrightarrow \begin{cases} \forall t \geq A & f(t) + \epsilon I_{k_{max}}(f) \geq 0, \\ \forall t \leq A & f(t) \geq 0, \\ on\ -A \leq t \leq A & f \text{ is decreasing }, \\ \forall t \leq -A & f(t) \geq f(-A), \\ \forall k \in [-2\frac{A}{\delta}, 2\frac{A}{\delta}] & I_{k-1}(f) \leq \frac{1}{\epsilon_k} I_k(f), \\ \forall t \leq -A & f(t) \leq \frac{1}{\epsilon'} I_{k_{min}}(f), \end{cases}$$

satisfies that, for any i, T_i is $d_{\mathcal{C}}$ contracting.

This cone may seem a bit artificial, however it behaves nicely with respect to the iteration of T_i. For the proof we need the following

Lemma 42. *Let $y, x > 0$, K linear and $a, b, u, v \geq 0$ such that $Kx \geq ax + uy$ and $Ky \leq by + vx$. If $a > b$ or $u > 0$, then there exist $\epsilon > 0$ such that if $\frac{1}{\epsilon}x \geq y$ then $\frac{1}{\epsilon}Kx > Ky$.*

Proof. If $a > b$, then for ϵ small enough, $\frac{b}{\epsilon} + v < \frac{a}{\epsilon}$ and we have $Ky \leq by + vx \leq (\frac{b}{\epsilon} + v)x < \frac{a}{\epsilon}x \leq \frac{1}{\epsilon}Kx$. If $u > 0$, then we have $\frac{1}{\epsilon}Kx - Ky \geq (\frac{a}{\epsilon} - v)x - (b - \frac{u}{\epsilon})y > 0$ for ϵ small enough. □

We can now carry on the proof of Proposition 41.

Proof. We construct the ϵ_k recursively. Because f is decreasing and e^{-U_i} are uniformly integrable, there exists u such that $\int_{-\alpha}^{\infty} f e^{-U_i(s)} ds \leq u I_{k_{min}}(f)$. We then have

$$\sup T_i f = \int_{\mathbb{R}} e^{-U_i(s)} f(s) ds \leq \sup f \cdot \int_{-\infty}^{-A} e^{-U_i(s)} ds + u I_{k_{min}}(f).$$

Moreover $I_{k_{min}}(Tf) \geq \delta e^{-U_i(-A)} I_{k_{min}}(Tf)$ and then $\int_{-\infty}^{-A} e^{-U_i(s)} ds < \delta e^{-U_i(-A)}$. By Lemma 42 there exists ϵ' such that for any i and t ,$T_i f(t) < \frac{1}{\epsilon'} I_{k_{min}}(T_i f)$.

Suppose we have constructed every ϵ_k up to $k = k_0$, and let us construct ϵ_{k_0+1}. Because of the induction hypothesis there exist b_{k_0} such that $\sup_t f(t) \leq b_{k_0} I_{k_0}(f)$ so there exists b'_{k_0} such that $I_{k_0}(T_i f) \leq b'_{k_0} I_{k_0}(f)$. Moreover

$$\begin{aligned} \forall a \in I_{k_0+1}, \quad T_i f(a) &= \int_{a-\tilde{x}_{i+1}-\tilde{x}_i}^{\infty} f(s) e^{-U_i(s)} ds \\ &\geq \int_{I_{k_0}} f(s) e^{-\max_{s \in I_{k_0}} U_i(s)} ds \\ &\geq e^{-\max_{s \in I_{k_0}} U_i(s)} I_{k_0}(f). \end{aligned}$$

So thanks to Lemma 42, there exists ϵ_{k_0+1} such that if for all $k \leq k_0, I_k(f) \leq \frac{1}{\epsilon_{k+1}} I_{k+1}(f)$ then for all $k \leq k_0 + 1, I_k(T_i f) < \frac{1}{\epsilon_{k+1}} I_{k_0}(T_i f)$. We also have for any $a \leq A$

$$\begin{aligned} T_i f(a) &= \int_{a-\tilde{x}_{i+1}-\tilde{x}_i}^{\infty} f(s) e^{-U(s)} ds \\ &\geq \int_{A-\delta}^{A} f(s) e^{-U_i(s)} ds - \epsilon \int_{A}^{\infty} e^{-U_i(s)} ds \cdot I_{k_{max}}(f) \\ &\geq e^{-U_i(A)} (1 - \epsilon \frac{\delta}{2}) I_{k_{max}}(f). \end{aligned}$$

In particular $T_i f(a) \geq 0$. Moreover for any $a > A$, we have

$$\begin{aligned} T_i f(a) \geq -\epsilon \int_{A}^{\infty} e^{-U_i(s)} ds \cdot I_{k_{max}}(f) &\geq -\epsilon \frac{\delta}{2} e^{-U_i(A)} I_{k_{max}}(f) \\ &\geq -\epsilon \frac{1}{1 - \epsilon \frac{\delta}{2}} I_{k_{max}}(Kf). \end{aligned}$$

Because $f \geq 0$ on $(-\infty, A]$, Tf is decreasing on $]-\infty, A+\delta]$. To conclude, it will be enough to compare $T_i f$ with $T_i g$ for $f, g \in \mathcal{C}$ and $I_{k_{max}}(f) = I_{k_{max}}(g) = 1$. Because all the inequalities become strict, there exists ϵ'' such that for any $f \in \mathcal{C}$ with $I_{k_{max}}(f) = 1$, if $\|g\|_{L^\infty} \leq \epsilon''$ then $T_i(f - g) \in \mathcal{C}$. Moreover for any $g \in \mathcal{C}$ with $I_{k_{max}}(g) = 1$, $\|g\|_{L^\infty} \leq \prod \frac{1}{\epsilon_k}$. So $T_i(f - \epsilon'' \prod \epsilon_k g) \in \mathcal{C}$. And this concludes the proof because then $\Delta \leq 2 \log(\epsilon'' \prod \epsilon_k)$. □

Remark 43. If we denote by $\mathcal{H}_k$ the assertion "$I_{k-1}(f) \leq \frac{1}{\epsilon_k} I_k(f)$", we have actually proved that if f satisfies all the condition of $\mathcal{C}$ except $(\mathcal{H}_i)_{i=r,...,k_{max}}$, then $T_i f$ satisfies all the condition of $\mathcal{C}$ except $(\mathcal{H}_i)_{i=r+1,...,k_{max}}$. This implies that if $f \geq 0$ and $\mathrm{supp}(f) \subset [-A, A]$, then $\prod_{i=k}^{k+n} T_i f \in \mathcal{C}$ is as wanted.

4.1.2 Decay of correlation.

Here we prove Theorem 4.

We can carry on with the construction of conditions like $I_{k-1}(f) \leq \frac{1}{\epsilon_k} I_k(f)$ after k_{max} in Proposition 41. We denote by $\mathcal{C}_m$ this more specified cone replacing $\forall k \in [-2\frac{A}{\delta}, 2\frac{A}{\delta}] I_{k-1}(f) \leq \frac{1}{\epsilon_k} I_k(f)$ by $\forall k \in [-2\frac{A}{\delta}, 2\frac{A}{\delta} + m] I_{k-1}(f) \leq \frac{1}{\epsilon_k} I_k(f)$. We have then the

Proposition 44. *If $g \in \mathcal{C}$ then $\prod_{i=k}^{k+n} T_i g \in \mathcal{C}_i$.*

Proof. By Proposition 41, $\prod_{i=k}^{k+n} T_i g \in \mathcal{C}$. Therefore it is enough to prove the remaining conditions. Let $g_i \in \mathcal{C}_i$. As previously,

$$T_i g(a) \geq -\epsilon \frac{\delta}{2} e^{-U_i(A+\delta i)} I_{k_{max}}(f) \geq -\epsilon \frac{1}{1 - \epsilon \frac{\delta}{2}} I_{k_{max}+i}(Tf).$$

As a consequence we have that for $f \geq 0$ and $\mathrm{supp}(f) \in (-\infty, A + n\delta]$ then $f \prod_{i=k}^{k+n} T_i g \geq 0$ for all $g \in \mathcal{C}$. □

Remark 45. If $f \geq 0$ and $\mathrm{supp}(f) \in [A, \infty)$ then $T_i f \in \mathcal{C}$.Therefore for f, $\mathrm{supp}(f) \in [a - dl, a - dl]$ $dl \leq \delta$ and $A \leq a \leq A + n\delta$ then $T_{n+k+1} F \prod_{i=k}^{k+n} T_i$ is order preserving for the cone $\mathcal{C}$.

Proposition 46. *For $f \geq 0$ with supp$(f) \in [a-dl, a-dl]$ $dl \leq \delta$ and $a \geq A+n\delta$, we have*

$$\left\langle 1, \prod_{i=k}^{k+n} T_i f \right\rangle \leq e^{-n(|\frac{a}{\delta}|^2 - c)} \langle 1, f \rangle \left\langle 1, \prod_{i=k}^{k+n} T_i 1 \right\rangle.$$

By iteration supp$(\prod_i Tf) \in [a - n \max(\tilde{a}_i - \tilde{a}_{i+1}), \infty)$. Therefore

$$\begin{aligned} \| \prod T_i f \|_\infty &\leq \langle 1, f \rangle e^{-\sum \min_i (U_i(a - k(\max(\tilde{a}_i - \tilde{a}_{i+1}))))} \\ &= \langle 1, f \rangle e^{-\gamma_{min} \sum (a - k(\max(\tilde{a}_i - \tilde{a}_{i+1}))^2}. \end{aligned}$$

Proof. There exists $\lambda > 0$ such that $\langle 1, \prod_{i=k}^{k+n-1} T_i 1 \rangle \geq \lambda^n$, and we set $c = \log \lambda$. The result follows. □

We are now ready to prove the decay of the correlation functions. Recall that for $i_1, i_2, \cdots, i_k$, we have the k-th marginal defined by

$$\rho_k(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) = \frac{1}{\mathcal{Z}_N(\beta)} e^{-\beta E(\tilde{x}_1, \ldots, \tilde{x}_N)}$$
$$\int \cdots \int_{-L < x_1 < x_2 < \ldots < x_N < L} \prod e^{-2\beta q_i \int_{\tilde{x}_i}^{x_i} \rho(y)(y - x_i) dy} \prod_{i \neq i_1, i_2, \ldots, i_N} dx_i.$$

Note that there exists r such that for $\rho_k(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \leq e^{-r \max |x|^3}$.

Corollary 47. *There exists $\kappa < 1$ and $C_k > 0$ such that*

$$|\rho_k(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) - \prod_{l=1}^{k} \rho_1(x_{i_l})| \leq C_k \kappa^{\inf |i_l - i_{l+1}|}.$$

Proof. Let $x_{i_1}, \cdots, x_{i_k} \in \mathbb{R}^k$ and let $\delta^{(n)}$ be an approximation of the Dirac δ_0. We evaluate

$$\begin{aligned}&\iiiint [\rho_k(y_{i_1}, y_{i_2}, \ldots, y_{i_k}) - \prod \rho_1(y_{i_1})] \prod \delta^{(n)}(y_{i_k} - x_{i_k}) dy_{i_k} \\ &= \iiiint [\rho_k(y_{i_1}, y_{i_2}, \ldots, y_{i_k}) - \prod \rho_1(y_{i_1})] \prod \delta^{(n)}_{x_{i_k}}(y_{i_k}) dy_{i_k}\end{aligned}$$

which, in our formalism, is equal to

$$\begin{aligned}&\frac{1}{\mathcal{Z}_N(\beta)}(u, T_N \ldots T_{K_k+1} \delta^{(n)}_{x_k} T_{K_k} \ldots T_{K_1+1} \delta^{(n)}_1 T_{K_1} \ldots T_0 v) \\ &- \prod \frac{1}{\mathrm{Z}}(u, T_N \ldots T_{K_1+1} \delta^{(n)}_{x_k} T_{K_1} \ldots T_0 v).\end{aligned}$$

To begin with, assume $-A < x_1, \cdots, x_k < A$. Because of Remark 45, for any $i \in [1, k]$, $T_{K_k+m} \ldots T_{K_k+1} \delta^{(n)}_{x_k} T_{K_k}$ is a positive operator. Changing C_k, we can suppose $\inf(|i_l - i_{l+1}|) > m$. Therefore, denoting $X_l = T_{K_l+m} \ldots T_{K_l+1} \delta^{(n)}_{x_l} T_{K_l}$, we can apply Theorem 33 and we obtain :

$$\begin{aligned}&|\frac{1}{\mathcal{Z}_N(\beta)}(u, T_N \ldots T_{K_k+1} \delta^{(n)}_{x_k} T_{K_k} \ldots T_{K_1+1} \delta^{(n)}_1 T_{K_1} \ldots T_0 v) \\ &- \prod \frac{1}{\mathrm{Z}}(u, T_N \ldots T_{K_1+1} \delta^{(n)}_{x_k} T_{K_1} \ldots T_0 v)| \\ &\leq C_k \kappa^{\inf(i_l - i_{l+1})},\end{aligned}$$

where $C_k = 2k(2k+1)R\kappa^{-m}$ if $\inf(i_l - i_{l+1})$ is larger that a constant c. Suppose that there exist $|x_i| > \epsilon \inf(|i_l - i_{l+1}|\delta)$ then $\rho_k(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \leq e^{-r(\epsilon\delta)^2 \inf_l |i_l - i_{l+1}|^2}$ and we are done. If for all $i, |x_i| \leq \epsilon \inf(|i_l - i_{l+1}|\delta)$, then as previously $T_{K_k+m} \ldots T_{K_k+1} \delta^{(n)}_{x_k} T_{K_k}$ is positive for $m = \epsilon \inf |i_l - i_{l+1}|$, hence

$$\begin{aligned}&|\frac{1}{\mathcal{Z}_N(\beta)}(u, T_N \ldots T_{K_k+1} \delta^{(n)}_{x_k} T_{K_k} \ldots T_{K_1+1} \delta^{(n)}_1 T_{K_1} \ldots T_0 v) \\ &- \prod \frac{1}{\mathrm{Z}}(u, T_N \ldots T_{K_1+1} \delta^{(n)}_{x_k} T_{K_1} \ldots T_0 v)| \\ &\leq C'_k \kappa^{(1-\epsilon) \inf(i_l - i_{l+1})}\end{aligned}$$

with $C_k = 2k(2k+1)R$. We can conclude replacing κ by $\kappa^{(1-\epsilon)}$. □

4.1.3 Smoothness of the free energy for the classical Jellium model.

Now we use Theorem 23 to prove the smoothness of the free energy. We first have to check its hypothesis. This is the aim of the following proposition

Proposition 48. *Let $u_0 \in T_i(\mathcal{C})$. Then*

$$\|[\partial_\beta^n T_i(\beta)]\|_{\mathcal{N}_i \to \mathcal{N}_{i+1}} < \infty$$

for all n, where $\mathcal{N}_i$ and $\mathcal{N}_{i+1}$ are the norm constructed in Proposition 22 around u_0 and $T_i(\beta)u_0$ respectively, and

$$\mathcal{N}(\partial_\beta^n T_i(\beta) u_0) < \infty.$$

To simplify the calculation, we introduce an approximating norm.We define

$$a_1(s) = \sup_{t \geq A} \frac{s(t) + \epsilon I_{k_{max}}(s)}{\nu},$$

$$a_2(s) = \sup_{t \leq A} \frac{s(t)}{\nu},$$

$$a_3(s) = \sup_{-A \leq t \leq A} \frac{s'(t)}{\nu},$$

$$a_4(s) = \sup_{t \leq -A} \frac{s(t) - s(-A)}{\nu},$$

$$a_5(s) = \sup_{k} \frac{\frac{1}{\epsilon_k} I_k(s) - I_{k-1}(s)}{\nu},$$

$$a_6(s) = \sup_{t \leq -A} \frac{s(t) - \frac{1}{\epsilon'} I_{k_{min}}(s)}{\nu},$$

and we set $A_\nu(s) = \max(a_1(s), a_2(s), a_3(s), a_4(s), a_5(s), a_6(s))$. Finally we define $\|s\|_\nu = \max(A(-s), A(s))$.

Proposition 49. *For u_0 in $T_i(\mathcal{C})$, there exists $\nu > 0$ such that*

$$\|.\|_{i+1} \leq \|.\|_\nu.$$

Proof. Let $u_0 \in T_i(\mathcal{C})$ We calculate the norm of Proposition 22. Let s be such that

$$\alpha u_0 \leq u_0 + s \leq \beta u_0$$

with

$$\alpha = \max_\alpha \begin{cases} \forall t \geq A & s(t) + \epsilon I_{k_{max}}(s) \geq (\alpha - 1)[u_0(t) + \epsilon I_{k_{max}}(t)], \\ \forall t \leq A & s(t) \geq (\alpha - 1) u_0(t), \\ \text{on } -A \leq t \leq A & s'(t) \leq (\alpha - 1) u_0'(t), \\ \forall t \leq -A & s(t) - s(-A) \geq (\alpha - 1)[u_0(t) - u_0(-A)], \\ \forall k \in [-2\frac{A}{\delta}, 2\frac{A}{\delta}] & I_{k-1}(s) - \frac{1}{\epsilon_k} I_k(s) \leq (1 - \alpha)[I_{k-1}(u_0) - \frac{1}{\epsilon_k} I_k(u_0)], \\ \forall t \leq -A & s(t) - \frac{1}{\epsilon'} I_{k_{min}}(f) \leq (1 - \alpha)[u_0(t) - \frac{1}{\epsilon'} I_{k_{min}}(u_0)]. \end{cases}$$

Because $u_0 \in T_i(\mathcal{C})$, there exists $\epsilon_0 > 0$ such that all the functions depending on u_0 on the right side of the equation $([u_0(t) + \epsilon I_{k_{max}}(t)], u_0(t), \cdots)$ can be bounded by ϵ_0. We obtain

$$\alpha' = \max_{\alpha'} \begin{cases} \forall t \geq A & s(t) + \epsilon I_{k_{max}}(s) \geq (\alpha' - 1)\epsilon_0 \\ \forall t \leq A & s(t) \geq (\alpha' - 1)\epsilon_0 \\ \text{on } -A \leq t \leq A & s'(t) \leq (\alpha' - 1)\epsilon_0 \\ \forall t \leq -A & s(t) - s(-A) \geq (\alpha' - 1)\epsilon_0 \\ \forall k \in [-2\frac{A}{\delta}, 2\frac{A}{\delta}] & I_{k-1}(s) - \frac{1}{\epsilon_k} I_k(s) \leq (1 - \alpha')\epsilon_0 \\ \forall t \leq -A & s(t) - \frac{1}{\epsilon'} I_{k_{min}}(s) \leq (1 - \alpha')\epsilon_0 \end{cases}$$

and we then have $\alpha' \leq \alpha$. We have constructed then $\|\|\|_\nu$ with $\nu = \epsilon_0$. □

We finish the proof of Proposition 48.

Proof. We can now calculate $\|\partial_\beta^n T_i(\beta)\|_{\|\|\|_i \to \|\|\|_\nu}$. Let w with $\|w\|_i \leq 1$, In particular, there exists $r > 0$ such that $d(u_0, u_0 + r.w) \leq 2r$. Therefore

$$(1 - 2r)u_0(t) \leq u_0 + rw(t) \leq (1 + 2r)u_0(t)$$

for all $t < A$. Hence $|I_{k_{max}}(r.w)| \leq 2r.I_{k_{max}}(u_0)$ and for all $t > A$:

$$(1 - 2r)[u_0(t)] - 4r.I_{k_{max}}(u_0) \leq u_0(t) + rw(t) \leq (1 + 2r)u_0(t) + 4r.I_{k_{max}}(u_0).$$

Therefore

$$[\partial_\beta^n T_i(\beta)w](x) = \int_{x - \tilde{x}_i - \tilde{x}_{i+1}}^{\infty} U_i(y)^n e^{-\beta U_i(y)} w(y) dy$$

and there exist c_1, c_2 and c_3 such that

$$\|[\partial_\beta^n T_i(\beta)w]\|_{L^\infty} \leq c_1 \|u_0\|_{L^\infty},$$

$$\|[\partial_\beta^n T_i(\beta)w]\|_{L^\infty} \leq c_2 \|u_0\|_{L^\infty}$$

and

$$\|[\partial_\beta^n T_i(\beta)w]'\|_{[-A,A]} \leq c_3 \|u_0\|_{L^\infty}.$$

There exists then c' such that $\|[\partial_\beta^n T_i(\beta)w]\|_\nu \leq c' \|u_0\|_{L^\infty}$. Then we have shown that $\|[\partial_\beta^n T_i(\beta)w]\|_{L^\infty}$ is a uniformly bounded operator for $\|\|\|_i \to \|\|\|_\nu$ and then for $\|\|\|_i \to \|\|\|_{i+1}$.We can now conclude the proof of Theorem 5. Moreover,

$$\begin{aligned} \int_{x - \tilde{x}_i - \tilde{x}_{i+1}}^{\infty} U_i(y)^n e^{-\beta U_i(y)} w(y) dy &\leq \int_{x - \tilde{x}_i - \tilde{x}_{i+1}}^{\infty} (Ay^2)^n e^{-\beta a y^2} w(y) dy \\ &\leq \|w\|_{L^\infty} (\frac{A}{\beta a})^n \Gamma(n - \frac{1}{2}) \end{aligned}$$

We can then apply Theorem 24 which ends the proof of analyticity of the free energy of the classical Jellium model. □

4.2 Proof for the quantum Jellium model

4.2.1 Decay of correlations, smoothness of the free energy.

Definition 50. For any f we define

$$T_i f(\gamma) = \int_E 1_{\forall t, \gamma(t) < \eta(t) + \delta_i} f(\eta) \nu_i(d\eta),$$

where $\nu_i = \frac{1}{c} \int_{\mathbb{R}} \nu_{i,xx} dx$ with Radon Nikodym density $\frac{d\nu_{i,xx}}{d\mu_{xx}}(\gamma) = e^{-\int_0^\beta U_i(\gamma(t))dt}$.

We recall the result concerning the homogeneous case.

Theorem 51. *T_i is a compact operator on $L^1((1+x^2)^{-1}dx)$ with a unique largest eigenvalue $\lambda_M > 0$.*

For the reader's convenience, we have written again the proof.

Proof. T_i is a compact operator. Indeed let $u \in L^1((1+x^2)^{-1}dx)$, then Tu is bounded, with finite variation. T is Hilbert-Smicht:

$$\iint_{E \times E} e^{-2\int U(\gamma(t))dt} 1_{\forall t, \gamma(t) \le \eta(t) + \delta_i} d\nu(\gamma) d\nu(\eta) < \infty.$$

We have to check that ν_i are bounded measures $\mathbb{P}(|B_t| > y) \le 2e^{-|y|^2}$. Then

$$\begin{aligned}\iint_{\mathbb{R}\times\Gamma} e^{-\int U_i(\gamma)dt} d\mu_{xx}(\gamma)dx &\le \int_{\mathbb{R}} [\mu_{xx}(\inf_t \gamma(t) < \frac{x}{2})] + e^{-\beta U_i(\frac{x}{2})} dx \\ &\le \int_{\mathbb{R}} c[e^{-(\frac{x}{2})^2} + e^{-\beta U(\frac{x}{2})}] dx < \infty\end{aligned}$$

The largest eigenvalue is unique because the operator is irreducible and we can apply the Krein Rutmann Theorem. □

Theorem 52. *Let T be a bounded real operator whose spectral radius $\rho(T)$ is a non degenerate eigenvalue with eigenvector u_0. Assume in addition that $T' = T - \rho(T)u_0 u_0^*$ has a spectral radius $\rho(T') < \rho(T)$. Then there exists a cone such that the operator is contracting.*

Proof. We can suppose that the largest eigenvalue is 1 and let u_0 be its eigenvector. We construct

$$\mathcal{C} = \text{positive linear combinations of } \cup_n T^n \Big(B(u_0, \epsilon(1-\epsilon_n)) \Big)$$

with ϵ_n a strictly decreasing sequence. Because there exists N such that $(T - u_0 u_0^*)^n$ is contracting for $n \ge N$,

$$\mathcal{C} = \text{positive linear combinations of } \cup_{n \le N_1} T^n \Big(B(u_0, \epsilon(1-\epsilon_n)) \Big).$$

Indeed, let $x \in B(u_0, \epsilon(1-\epsilon_{N_1+1}))$, $x = u_0 + y + su_0$ with $u_0^* y = 0$ and $\|y\| \le c\|\epsilon\|$ and $s \le c\|\epsilon\|$. There exists N_1 such that $\|(T - u_0 u_0^*)^{N_1+1}\| \le \frac{(1-\epsilon_0)}{2c}$. Then

$$\begin{aligned} T^{N_1+1}(x) &= (u_0 u_0^*)x + (T - u_0 u_0^*)^{N_1+1}(x) \\ &= (1+s)u_0 + (T - u_0 u_0^*)^{N_1+1}(y) \in B((1+s)u_0, (1+s)\epsilon(1-\epsilon_0)), \end{aligned}$$

because $\|(T - u_0 u_0^*)^{N_1+1}(y)\| \le \frac{\epsilon c(1-\epsilon_0)}{2c} \le (1+s)\epsilon(1-\epsilon_0)$.

Let $x \in \mathcal{C}$, with $u_0^* x = 1$. Then $x = \sum_{i=0}^{N_1} a_i x_i$, $a_i \ge 0$ with $x_i \in T^i(B(u_0, \epsilon(1-\epsilon_i)))$. Let us construct α and β such that $\alpha u_0 \le T(x) \le \beta u_0$.

First because $B(u_0, \epsilon(1-\epsilon_0)) \subset \mathcal{C}$, we choose $\beta \le \frac{\|T(x)\|}{\epsilon(1-\epsilon_0)}$, and we immediatly have $u_0 - \frac{1}{\beta}T(x) \ge 0$.

Second for all i, $T(x_i) \in T^{i+1}(B(u_0, \epsilon(1-\epsilon_i)))$ and therefore $T(\frac{(1-\epsilon_{i+1})}{(1-\epsilon_i)}x_i) \in T^{i+1}(B(\frac{(1-\epsilon_{i+1})}{(1-\epsilon_i)}u_0, \epsilon(1-\epsilon_{i+1})))$. We have then

$$T(\frac{(1-\epsilon_{i+1})}{(1-\epsilon_i)}x_i) - (1 - \frac{(1-\epsilon_{i+1})}{(1-\epsilon_i)})u_0 \in T^{i+1}(B(u_0, \epsilon(1-\epsilon_{i+1})))$$

and also

$$T(\frac{(1-\epsilon_{i+1})}{(1-\epsilon_i)}x_i) \ge (1 - \frac{(1-\epsilon_{i+1})}{(1-\epsilon_i)})u_0.$$

So with $M = \max \frac{(1-\epsilon_{i+1})}{(1-\epsilon_i)}$ and $m = \min \frac{(1-\epsilon_{i+1})}{(1-\epsilon_i)}$ we have

$$MT(x) \ge m(\sum a_i)u_0$$

and we can conclude that

$$\frac{m \sum a_i}{M} u_0 \le T(x).$$

□

Such a construction is stable under small compact perturbations.

Proposition 53. *There exists $\delta_0 > 0$ such that for δT compact operator with $\|\delta T\| \le \delta_0$,*

$$\Delta_{\mathcal{C}}((T + \delta T)(\mathcal{C})) < 2\Delta_{\mathcal{C}}(T(\mathcal{C}))$$

where $\mathcal{C}$ is the cone constructed in Theorem 52

In particular $T + \delta T$ is a positive contracting operator for $d_{\mathcal{C}}$.

Proof. We rewrite the proof of Theorem 52. We keep u_0 because $(T+\delta T)(u_0) \in B(u_0, (\epsilon(1-\epsilon_0)))$. First, it is enough to change $\beta \le \frac{\|(T+\delta T)(x)\|}{\epsilon(1-\epsilon_0)}$ and we have

$u_0 - \frac{1}{\beta}(T+\delta T)(x) \geq 0$. Second, we also have

$$(T+\delta T)(\frac{(1-\epsilon_{i+1})}{(1-\epsilon_i)}x_i) - \delta T(\frac{(1-\epsilon_{i+1})}{(1-\epsilon_i)}x_i)$$
$$\geq (T+\delta T)(\frac{(1-\epsilon_{i+1})}{(1-\epsilon_i)}x_i) - u_0\frac{\|\delta T(\frac{(1-\epsilon_{i+1})}{(1-\epsilon_i)}x_i)\|}{\epsilon(1-\epsilon_0)}$$
$$\geq [(1-\frac{(1-\epsilon_{i+1})}{(1-\epsilon_i)}) - \frac{\delta_0}{\epsilon(1-\epsilon_0)}]u_0$$

and we can finish the proof as previously for δ_0 small enough. □

Remark 54. Actually we have that $T(x) \in \mathring{\mathcal{C}}$ for all $x \in \mathcal{C}$. Indeed, as previously

$$\frac{m\sum a_i}{M}u_0 \leq T(x)$$

and then for any $\|y\| \leq \frac{\epsilon(1-\epsilon_0)}{M}m\sum a_i$ we have $T(x) + y \geq \frac{m\sum a_i}{M}u_0 + y \geq 0$.

The construction of the cone is simple enough that we can calculate the norm of Theorem 22. Because of the previous remark it is equivalent to the space norm.

Proposition 55. *There exists $c > 0$ such that for any y in the projected space,*

$$c\|y\| \leq \|y\|_N \leq \frac{1}{c}\|y\|$$

where $\|\|_N$ is the norm constructed in Proposition 22 for the cone $\mathcal{C}$ in a neighborhood of $T(x)$.

Proof. Because of the previous remark, $T(x) \in \mathring{\mathcal{C}}$. Therefore there exists $r > 0$ such that $B(T(x), r) \subset \mathcal{C}$. For any y we have

$$T(x)\left(1 - \frac{s\|y\|}{r}\right) \leq T(x) + sy \leq T(x)\left(1 + \frac{s\|y\|}{r}\right)$$

and we obtain $d_{\mathcal{C}}(T(x)+sy, T(x)) \leq 2\frac{s\|y\|}{r} + o(\frac{s\|y\|}{r})$. For the other direction, $\mathcal{C} \subset$ cone from $B(u_0, \frac{1}{2})$ (for ϵ small). In addition $T(x) + y \notin \mathcal{C}$ for $\|y\| = 1$ and then $\alpha T(x) \leq T(x) + sy$ implies $\alpha \leq (1-s)$ and so $\|y\|_N \leq \|y\|$. □

We can now finish the proof of Theorem 10

Proof. [Theorem 10] $T(\beta)$ is C^∞ for the norm $\|\|$ and thanks to the previous proposition also for any norm constructed $\|\|_N$ around $T(\beta)(x)$. We can then apply Theorem 23. The analyticity follows as well: if $T(\beta)$ is analytic with coefficient bounded by r^n for the norm $\|\|$ then the coefficients are bounded by cr^n for the norm $\mathcal{N}$. □

We focus now on the decay of correlation and the proof of Theorem 9. Recall that we want to prove that there exists $\kappa < 1$ such that

$$|\rho_k(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) - \prod \rho_1(x_{i_1})| \leq C_k \kappa^{\min |i_l - i_{l+1}|}$$

Proof. [Theorem 9] There exists $\epsilon_0 > 0$ such that for f with $\|f\|_{L^1} \leq \epsilon_0$, $T_{i+1}(1+f)T_i$ are positive operator for the cone $\mathcal{C}$.

Indeed, let $u \in \mathcal{C}$, then $T_i(u) \in L^\infty(\Gamma)$, therefore $fT_i(u) \in L^1$ with $\|fT_i(u)\|_{L^1} \leq c\delta_0\|u\|$, and finally $\|T_{i+1}fT_i(u)\|_{L^\infty} \leq c'\delta\|u\|$. Because $T(\mathcal{C})$ is compact on the projected space and thanks to Remark 54, there exists $r > 0$ such that for all i, $u \in \mathcal{C}$, $B(T_{i+1}T_i(u), r\|u\|) \subset \mathcal{C}$. As a conclusion $T_{i+1}(1+f)T_i(u) = T_{i+1}T_i(u) + T_{i+1}fT_i(u) \in \mathcal{C}$. We can now apply Theorem 33 with the same notation. We find

$$\begin{aligned} &e^{-2(k+1)R(\sum_{j=0}^{k} \kappa^{K_{j+1}-K_j})} \prod_{i=1}^{k} \rho_{K_i}(T_{l_i+1}(1+f_i)T_{l_i}) \\ &\quad \leq \rho_{K_k,\ldots,K_1}(T_{l_k+1}(1+f_k)T_{l_k}, \ldots, T_{l_1}(1+f_1)T_{l_1}) \end{aligned}$$

and

$$\begin{aligned} &\rho_{K_k,\ldots,K_1}(T_{l_k+1}(1+f_k)T_{l_k}, \ldots, T_{l_1}(1+f_1)T_{l_1}) \\ &\quad \leq e^{2(k+1)R(\sum_{j=0}^{k} \kappa^{K_{j+1}-K_j})} \prod_{i=1}^{k} \rho_{K_i}(T_{l_i+1}(1+f_i)T_{l_i}). \end{aligned}$$

The rest follows from a induction on k and this concludes the proof of Theorem 9. □

A Proof of Proposition 3

If $I' = \{i_1, i_2\}$, then x_{i_1} and x_{i_2} are independent, $\rho_2(x_{i_1}, x_{i_2}) = \rho(x_{i_1})\rho(x_{i_2})$ and then $\rho^T(x_{i_1}, x_{i_2}) = 0$. For larger a I', we have

$$\begin{aligned} &\sum_{I_1 \cup I_2 \cup \ldots \cup I_r = I',\, l=1} \prod^{r} \rho^T_{|I_l|}((x_i)_{i \in I_l}) \\ &= \sum_{\substack{I_1 \cup I_2 \cup \ldots \cup I_r = \{1,\ldots,n\} \\ I_l \subset I \text{ or } I_l \subset J.}} \prod_{l=1}^{r} \rho^T_{|I_l|}((x_i)_{i \in I_l}) \\ &= \left(\sum_{I_1 \cup I_2 \cup \ldots \cup I_{r_1} = I \cap I'} \prod_{l=1}^{r_1} \rho^T_{|I_l|}((x_i)_{i \in I_l}) \right) \left(\sum_{I_1 \cup I_2 \cup \ldots \cup I_{r_2} = J \cap I'} \prod_{l=1}^{r_1} \rho^T_{|I_l|}((x_i)_{i \in I_l}) \right) \\ &= \rho_{|I' \cap I|}((x_i)_{i \in I \cap I'}) \rho_{|I' \cap J|}((x_i)_{i \in J \cap I'}) \end{aligned}$$

and therefore

$$\begin{aligned}
&\rho^T_{|I'|}((x_i)_{i\in I})\\
&= \rho_{|I'\cap I|}((x_i)_{i\in I\cap I'})\rho_{|I'\cap J|}((x_i)_{i\in J\cap I'}) - \sum_{I_1\cup I_2\cup\ldots\cup I_r = I',\, l=1} \prod^{r} \rho^T_{|I_l|}((x_i)_{i\in I_l})\\
&= 0.
\end{aligned}$$

References

[1] Michael Aizenman, Sabine Jansen, and Paul Jung. Symmetry Breaking in Quasi-1D Coulomb Systems. *Ann. Henri Poincaré*, 11(8):1453–1485, Dec 2010.

[2] Michael Aizenman and Philippe A Martin. Structure of Gibbs states of one dimensional Coulomb systems. *Communications in Mathematical Physics*, 78(1):99–116, 1980.

[3] Herbert Amann. Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. *SIAM review*, 18(4):620–709, 1976.

[4] R. J. Baxter. Statistical mechanics of a one-dimensional Coulomb system with a uniform charge background. *Proc. Cambridge Philos. Soc.*, 59:779–787, 1963.

[5] Garrett Birkhoff. Extensions of Jentzsch's theorem. *Transactions of the American Mathematical Society*, 85(1):219–227, 1957.

[6] Xavier Blanc and Mathieu Lewin. The Crystallization Conjecture: A Review. *EMS Surv. Math. Sci.*, 2(2):219–306, 2015.

[7] Jonathan M Borwein, Adrian Stephen Lewis, and Roger D Nussbaum. Entropy minimization, dad problems, and doubly stochastic kernels. *Journal of Functional Analysis*, 123(2):264–307, 1994.

[8] Matthias Brack. The physics of simple metal clusters: self-consistent Jellium model and semiclassical approaches. *Reviews of modern physics*, 65(3):677, 1993.

[9] HJ Brascamp and EH Lieb. Some inequalities for Gaussian measures and the long-range order of the one-dimensional plasma. In *Inequalities*, pages 403–416. Springer, 2002.

[10] Ph. Choquard. On the statistical mechanics of one-dimensional Coulomb systems. *Helv. Phys. Acta*, 48(4):585–598, 1975.

[11] Joel E Cohen. Ergodic theorems in demography. *Bulletin (New Series) of the American Mathematical Society*, 1(2):275–295, 1979.

[12] László Erdős. Universality for random matrices and log-gases. In *Current developments in mathematics 2012*, pages 59–132. Int. Press, Somerville, MA, 2013.

[13] Simon P Eveson and Roger D Nussbaum. An elementary proof of the birkhoff-hopf theorem. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 117, pages 31–55. Cambridge University Press, 1995.

[14] Peter J Forrester. *Log-gases and random matrices (LMS-34)*. Princeton University Press, 2010.

[15] G. Giuliani and G. Vignale. *Quantum Theory of the Electron Liquid*. Cambridge University Press, 2005.

[16] Ch. Gruber, Joel L. Lebowitz, and Ph. A. Martin. Sum rules for inhomogeneous Coulomb systems. *J. Chem. Phys.*, 75(2):944–954, 1981.

[17] Ch Gruber, Ch Lugrin, and Ph A Martin. Equilibrium equations for classical systems with long range forces and application to the one dimensional Coulomb gas. *Helv Phys. Acta*, 51(5-6):829–866, 1978.

[18] Yves Guivarc'h and Jean Hardy. Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d'Anosov. 24(1):73–98, 1988.

[19] Peter Hall and Christopher C Heyde. *Martingale limit theory and its application*. Academic press, 2014.

[20] Loïc Hervé. Vitesse de convergence dans le théorème limite central pour des chaînes de Markov fortement ergodiques. *Ann. Inst. Henri Poincaré Probab. Stat.*, 44(2):280–292, 2008.

[21] P. Hohenberg and W. Kohn. Inhomogeneous electron gas. *Phys. Rev.*, 136(3B):B864–B871, Nov 1964.

[22] Eberhard Hopf. An inequality for positive linear integral operators. *J. Math. Mech.*, 12:683–692, 1963.

[23] S. Jansen and P. Jung. Wigner crystallization in the quantum 1d jellium at all densities. *Comm. Math. Phys.*, pages 1–22, 2014.

[24] S. Jansen, E. H. Lieb, and R. Seiler. Symmetry breaking in Laughlin's state on a cylinder. *Comm. Math. Phys.*, 285(2):503–535, Jan 2009.

[25] Sabine Jansen and Paul Jung. Wigner crystallization in the quantum 1d Jellium at all densities. *Communications in Mathematical Physics*, 331(3):1133–1154, 2014.

[26] H Kunz. The one-dimensional classical electron gas. *Annals of Physics*, 85(2):303 – 335, 1974.

[27] Émile Le Page. Théoremes limites pour les produits de matrices aléatoires. In *Probability measures on groups*, pages 258–303. Springer, 1982.

[28] Mathieu Lewin, Elliott H. Lieb, and Robert Seiringer. Statistical Mechanics of the Uniform Electron Gas. *J. Éc. polytech. Math.*, 5:79–116, 2018.

[29] S. Lundqvist and N.H. March, editors. *Theory of the Inhomogeneous Electron Gas*. Physics of Solids and Liquids. Springer US, 1983.

[30] R.G. Parr and W. Yang. *Density-Functional Theory of Atoms and Molecules*. International Series of Monographs on Chemistry. Oxford University Press, USA, 1994.

[31] Pierre Pfeuty. The one-dimensional Ising model with a transverse field. *Annals of Physics*, 57(1):79–90, 1970.

[32] David Ruelle. *Statistical mechanics: Rigorous results*. World Scientific, 1999.

[33] Eugene Wigner. On the interaction of electrons in metals. *Physical Review*, 46(11):1002, 1934.